

LESSON 12 9709 TO 9231

Confidence intervals foundations

Estimate a population mean while showing how much uncertainty remains after sampling.

By the end of this lesson

- Calculate and interpret a normal confidence interval for a population mean.
- Distinguish exact known-variance intervals from approximate intervals.
- Explain how confidence, variability and sample size affect width.
- Find a minimum sample size for a target margin of error.

Before you begin: Distinguish a population mean from an observed sample mean; Calculate known and estimated standard errors; Use central areas of the standard normal distribution.

1. A sample estimate comes with uncertainty

The observed sample mean $\bar{x}$ is a point estimate of the population mean μ . Another independent sample will usually give a different mean. A confidence interval combines the point estimate with a margin of error to express this sampling uncertainty.

KEY FORMULA Structure of an interval

$$\text{estimate} \pm \text{margin of error}$$

The two endpoints are the estimate minus the margin and the estimate plus the margin. The full width is twice the margin.

The population mean is fixed but unknown. Before sampling, $\bar{X}$ and the resulting interval are random because they depend on the sample. Once the data are observed, $\bar{x}$ and the interval endpoints are fixed numbers. This distinction explains the repeated-sampling meaning of confidence.

2. A normal interval with known population variance

KEY FORMULA Confidence interval for a population mean

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$$

Use a random independent sample and a known population standard deviation σ . The population must be normal, or a suitable large-sample normal approximation must apply.

Here n is the sample size, $\sigma/\sqrt{n}$ is the standard error of the sample mean, and z^* is a standard normal critical value chosen for the confidence level. The margin of error is z^* times the standard error.

For a normal population with known variance, the normal method is exact for any sample size, apart from rounding critical values. For a non-normal population with finite variance, a suitable central limit theorem approximation gives approximate coverage. A sample size of 30 is not a universal guarantee.

KEY FORMULA Why the 95% method works

$$P\left(\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

Under the normal sampling model, this follows by rearranging $-1.96 \leq (\bar{X} - \mu)/(\sigma/\sqrt{n}) \leq 1.96$. The approximation here reflects the rounded critical value.

3. Choose the central area

A two-sided confidence interval leaves equal probabilities in the two tails. For 95% confidence, the central area is 0.95 and each tail has area 0.025. A larger confidence level requires a larger critical value.

Confidence level	Probability in each tail	z^*
90%	0.05	1.645
95%	0.025	1.96
99%	0.005	2.576

METHOD TIP Match the interval to the confidence level

Do not use 1.645 for a two-sided 95% interval: it leaves 5% in each tail and therefore gives 90% central coverage.

4. Calculate and interpret an interval

WORKED EXAMPLE The population mean fill volume

Fill volumes are normally distributed with known standard deviation 12 ml. An independent random sample of 64 containers has mean 502 ml. Calculate a 95% confidence interval for the population mean.

1. Calculate the standard error of the sample mean.

$$SE = \frac{12}{\sqrt{64}} = 1.5 \text{ ml}$$

2. Multiply by the 95% critical value.

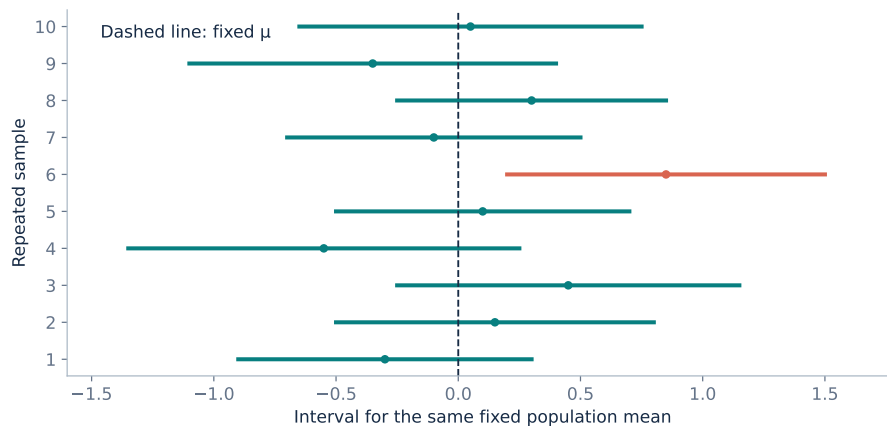
$$\text{margin} = 1.96(1.5) = 2.94 \text{ ml}$$

3. Subtract and add the margin to the observed mean.

$$502 \pm 2.94 = (499.06, 504.94) \text{ ml}$$

Answer. We are 95% confident that the population mean fill volume lies between 499.06 ml and 504.94 ml.

The confidence statement refers to the procedure: if we repeatedly took samples in the same way and constructed an interval each time, about 95% of those intervals would contain the fixed population mean. Some intervals miss it. We cannot tell from one sample alone whether our particular interval is one of them.



Illustrative intervals from repeated samples surround different sample means. The population mean is fixed.

The diagram explains coverage; its small collection of intervals is not intended to contain exactly 95% successful intervals.

5. Interval width and sample size

KEY FORMULA Width and margin

$$\text{width} = \frac{2z^*\sigma}{\sqrt{n}}, \quad \text{margin} = \frac{z^*\sigma}{\sqrt{n}}$$

These expressions use a known population standard deviation and the normal interval above.

- Higher confidence increases the critical value, producing a wider interval.
- Greater population variability increases the width.
- A larger sample reduces the width through the square root of n .
- Multiplying sample size by four halves the margin when confidence and population variability stay fixed.

To plan a sample, rearrange the margin formula and round the required sample size upwards. Rounding to the nearest integer can fail to meet the target. If the question specifies the full width, halve it first to obtain the margin.

WORKED EXAMPLE Planning a sufficiently precise estimate

A population has known standard deviation 10. Find the smallest sample size giving a 95% normal confidence interval with margin at most 2. Assume independent random sampling and an appropriate normal sampling model.

1. Write the required bound on the margin.

$$\frac{1.96(10)}{\sqrt{n}} \leq 2$$

2. Rearrange, using positive quantities.

$$n \geq \left(\frac{1.96(10)}{2} \right)^2 = 96.04$$

3. The sample size must be a whole number at least 96.04, so round upwards.

Answer. The smallest suitable sample size is $n = 97$.

6. An approximate interval using an estimated variance

If the population standard deviation is unknown, estimate it using the sample standard deviation s , where s^2 uses denominator $n - 1$. For a suitable large sample, the estimated standard error $s/\sqrt{n}$ can be used in an approximate normal interval. The additional estimation means the normal coverage is no longer exact.

KEY FORMULA Large-sample normal interval

$$\bar{x} \pm z^* \frac{s}{\sqrt{n}}$$

Use independent random observations and a justified large-sample normal approximation. A small normal sample with unknown variance requires a t interval, studied in Further Statistics.

WORKED EXAMPLE A mean with unknown population variance

An independent random sample of 100 observations gives $\bar{x} = 80$ and $s = 15$. A large-sample normal approximation is suitable. Find an approximate 95% confidence interval for μ .

1. Estimate the standard error.

$$\widehat{SE} = \frac{15}{\sqrt{100}} = 1.5$$

2. Calculate the approximate margin and interval.

$$80 \pm 1.96(1.5) = 80 \pm 2.94 = (77.06, 82.94)$$

Answer. An approximate 95% confidence interval for the population mean is (77.06, 82.94).

7. Avoid three interpretation mistakes

COMMON MISTAKE Confusing a mean with individual observations

A 95% confidence interval for μ does not contain 95% of individual observations. Its purpose is to estimate the population mean; an interval for individual values addresses a different question.

COMMON MISTAKE Treating the fixed mean as random

In this repeated-sampling framework, do not describe a calculated interval as assigning probability 0.95 to the fixed parameter. Explain that the interval method captures the true mean in about 95% of repeated samples.

COMMON MISTAKE Assuming a narrow interval removes every problem

The margin reflects sampling variability under the model. It does not automatically account for biased sampling, measurement problems or failures of independence.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

A normal population has known standard deviation 9. An independent random sample of 36 has mean 42. Find a 90% confidence interval for the population mean.

1. Use the 90% critical value and calculate the standard error.

$$SE = 9/\sqrt{36} = 1.5, \quad z^* = 1.645$$

2. Calculate the margin and endpoints.

$$\text{margin} = 1.645(1.5) = 2.4675, \quad 42 \pm 2.4675 = (39.5325, 44.4675)$$

Answer. The 90% confidence interval is approximately (39.53, 44.47).

PRACTICE Question 2

A population has known standard deviation 6. Find the smallest sample size for a 99% normal confidence interval with margin at most 1. Assume independent random sampling and a suitable normal sampling model.

1. Write the margin inequality.

$$\frac{2.576(6)}{\sqrt{n}} \leq 1$$

2. Square and round upwards.

$$n \geq (2.576 \times 6)^2 = 238.887936$$

Answer. The smallest sample size is 239.

PRACTICE Question 3

A normal population has known standard deviation 8. An independent random sample of 64 has mean 25. Find a 95% confidence interval and the sample size needed to halve its margin at the same confidence level.

1. The standard error is one.

$$SE = 8/\sqrt{64} = 1, \quad 25 \pm 1.96 = (23.04, 26.96)$$

2. Halving the margin requires four times as many observations.

$$n_{\text{new}} = 4(64) = 256$$

Answer. The interval is (23.04, 26.96). A sample size of 256 halves the margin.

PRACTICE Question 4

An independent random sample has $n = 64$, $\sum x_i = 1920$ and $\sum x_i^2 = 61632$. Population variance is unknown, and a large-sample normal approximation is suitable. Find an approximate 95% confidence interval for the population mean.

1. Estimate the mean and population variance from the summaries.

$$\bar{x} = 1920/64 = 30, \quad s^2 = \frac{61632 - 1920^2/64}{63} = \frac{4032}{63} = 64$$

2. Use s , rather than s squared, in the estimated standard error.

$$s = 8, \quad \widehat{SE} = 8/\sqrt{64} = 1$$

3. Form the approximate interval.

$$30 \pm 1.96 = (28.04, 31.96)$$

Answer. An approximate 95% confidence interval for μ is (28.04, 31.96).

Ready to move on?

- ☐ I can choose the critical value for the stated confidence level.
- ☐ I can use the correct known or estimated standard error.
- ☐ I can interpret confidence through repeated sampling.
- ☐ I can distinguish margin of error from full interval width.
- ☐ I can rearrange a margin requirement and round the sample size upwards.

Next: Further Statistics 9231. Continue to t intervals and intervals for differences between population means, building on the same estimate-and-standard-error structure.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

[Cambridge 9709 syllabus, section 6.4: Sampling and estimation](#) · [Cambridge 9231 prior-learning guidance](#) · [Cambridge 9231 syllabus, section 4: Further Probability and Statistics](#)