

LESSON 11 9709 TO 9231

Hypothesis testing foundations

Use a sample mean to assess evidence about a population claim and state a careful conclusion.

By the end of this lesson

- Write null and alternative hypotheses about a population mean.
- Choose one or two tails from the research question.
- Carry out exact known-variance and suitable approximate normal mean tests.
- Interpret significance, critical regions and conclusions correctly.

Before you begin: Calculate the standard error of a sample mean; Use normal probabilities and standard normal critical values; Distinguish population parameters from sample estimates.

1. Compare the observation with a population claim

A hypothesis test starts with a claim about a population parameter. We temporarily assume that claim is true, then ask whether the observed sample result would be unusually extreme under that assumption. An extreme result gives evidence against the claim; an ordinary result provides insufficient evidence to reject it.

The null hypothesis, H_0 , supplies the reference value. For a population mean we write $H_0 : \mu = \mu_0$, where μ_0 is the claimed value. The alternative hypothesis, H_1 , describes the departure being investigated. Hypotheses concern μ , not the observed sample mean $\bar{x}$: the sample mean is evidence about the population.

METHOD TIP Choose the direction first

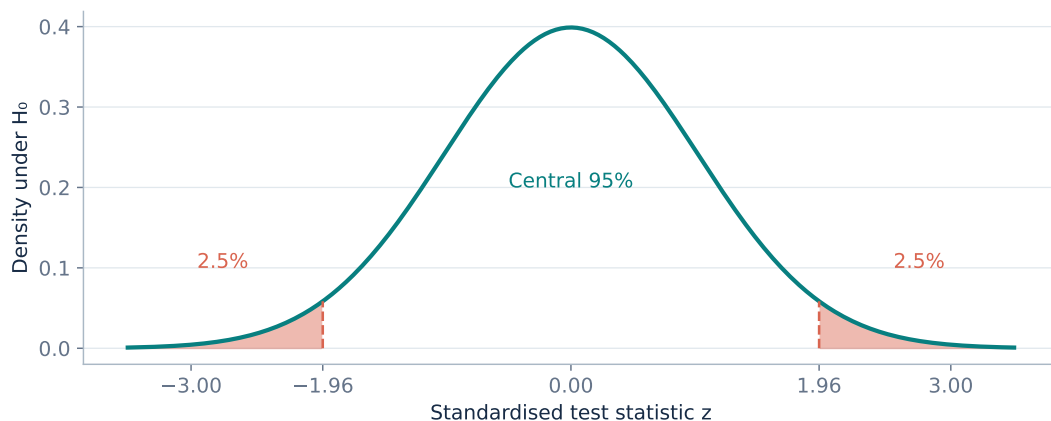
Select H_1 from the investigation before examining the sample result. Choosing the favourable tail after seeing the data changes the intended testing procedure.

2. Significance levels and critical regions

The critical region is the set of sufficiently extreme statistic values for which we reject H_0 . A significance level of 5% assigns total probability 0.05 to this region under the null model. For an approximate test, this probability is approximately 0.05. It is not the probability that H_0 is true.

Alternative	Tail or tails	5% normal critical region
$H_1 : \mu > \mu_0$	Upper	$z > 1.645$
$H_1 : \mu < \mu_0$	Lower	$z < -1.645$
$H_1 : \mu \neq \mu_0$	Two	$z < -1.96$ or $z > 1.96$

A two-tailed test looks for a departure in either direction. Split the significance level equally: at 5%, each tail contains 2.5%. The displayed cutoffs are rounded standard normal quantiles; an equality convention does not affect probabilities in a continuous model.



For a two-tailed normal test at 5%, the rejection regions lie beyond approximately -1.96 and 1.96 , with probability 0.025 in each tail under H_0 .

3. Standardise the sample mean under the null

KEY FORMULA Known population standard deviation

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}, \quad z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Z is the random statistic before sampling; z is its observed value. Under H_0 , $Z \sim N(0, 1)$ for an independent random sample from a normal population with known σ .

The denominator is the standard error of the sample mean, not the standard deviation of individual observations. A suitable large sample from a non-normal population with finite variance can justify an approximate normal test through the central limit theorem. There is no universal sample-size threshold that makes this approximation reliable.

- Check random sampling and independent observations.
- Identify whether the population standard deviation is known or estimated.
- State whether normality of the test statistic is exact or an approximation.

4. Worked tests with known variance

WORKED EXAMPLE Is a filling machine underfilling?

Bag masses are normal with known standard deviation 8 g. An independent random sample of 25 bags has mean 496.8 g. Test at 5% whether the population mean is less than 500 g.

1. Define μ as the population mean bag mass and choose a lower-tailed alternative.

$$H_0 : \mu = 500, \quad H_1 : \mu < 500$$

2. Calculate the standard error and observed statistic.

$$SE = \frac{8}{\sqrt{25}} = 1.6, \quad z = \frac{496.8 - 500}{1.6} = -2$$

3. The critical region is $z < -1.645$. Since $-2 < -1.645$, reject H_0 .

Answer. There is sufficient evidence at the 5% significance level that the population mean bag mass is below 500 g.

WORKED EXAMPLE Has a mean fill volume changed?

Bottle volumes are normal with known standard deviation 12 ml. An independent random sample of 36 bottles has mean 254 ml. Test at 5% whether the population mean differs from 250 ml.

1. A change in either direction requires two tails.

$$H_0 : \mu = 250, \quad H_1 : \mu \neq 250$$

2. Calculate the statistic using the standard error.

$$SE = \frac{12}{\sqrt{36}} = 2, \quad z = \frac{254 - 250}{2} = 2$$

3. At 5%, reject for $|z| > 1.96$. Here $2 > 1.96$.

Answer. Reject H_0 . There is sufficient evidence at 5% that the population mean volume differs from 250 ml.

5. A suitable large sample with unknown variance

When the population variance is unknown, estimate it using s^2 with denominator $n - 1$. For a suitable large sample, replacing σ by s gives an approximate normal test. This is not an exact known-variance test. Small normal samples with unknown variance require the t methods taught in Further Statistics.

KEY FORMULA Approximate statistic using an estimated standard error

$$z_{\text{approx}} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Use an independent random sample and a justified large-sample normal approximation. The sample standard deviation s estimates σ .

WORKED EXAMPLE A large sample of item masses

An independent random sample of 100 items gives $\bar{x} = 51.8$ g and $s = 10$ g. A large-sample normal approximation is suitable. Test at 5% whether the population mean exceeds 50 g.

1. State an upper-tailed alternative.

$$H_0 : \mu = 50, \quad H_1 : \mu > 50$$

2. Estimate the standard error.

$$\widehat{SE} = \frac{10}{\sqrt{100}} = 1, \quad z_{\text{approx}} = \frac{51.8 - 50}{1} = 1.8$$

3. Since $1.8 > 1.645$, the result falls in the approximate upper-tailed 5% critical region.

Answer. Reject H_0 . The approximate test gives sufficient evidence at 5% that the population mean mass exceeds 50 g.

6. What the conclusion does and does not say

If the statistic is in the critical region, reject H_0 and state the evidence for H_1 in context. Otherwise, do not reject H_0 : there is insufficient evidence for the proposed departure at that significance level. A non-significant result does not prove that the null value is correct.

In the bottle example, an observed mean of 253 ml would give $z = (253 - 250)/2 = 1.5$. This lies outside the two-tailed critical region. The conclusion would be insufficient evidence that the population mean differs from 250 ml, even though the sample mean itself differs from 250.

COMMON MISTAKE Statistical evidence is not practical size

A decision describes evidence under a model. It does not, by itself, tell you whether the departure is large enough to matter in practice.

7. Optional: expressing the evidence with a p-value

A p-value is the probability, assuming H_0 and the model assumptions, of a statistic at least as extreme as the observed value in the direction or directions specified by H_1 . Reject when the p-value is at most the significance level.

KEY FORMULA Normal p-values

$$p_{\text{upper}} = 1 - \Phi(z), \quad p_{\text{lower}} = \Phi(z), \quad p_{\text{two}} = 2[1 - \Phi(|z|)]$$

Here $\Phi(a) = P(Z \leq a)$ for $Z \sim N(0, 1)$. These formulas use standard normal symmetry.

For the bottle test, $p = 2P(Z > 2) \approx 0.0455 < 0.05$, agreeing with the critical-region decision. The p-value is not the probability that the null hypothesis is true.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

Measurements are normal with known $\sigma = 15$. An independent random sample of 36 gives $\bar{x} = 125$. Test at 1% whether the population mean exceeds 120. Use the upper-tail critical value 2.326.

1. State the hypotheses and calculate the standard error.

$$H_0 : \mu = 120, \quad H_1 : \mu > 120, \quad SE = 15/\sqrt{36} = 2.5$$

2. Calculate and compare the statistic.

$$z = \frac{125 - 120}{2.5} = 2 < 2.326$$

Answer. Do not reject H_0 . There is insufficient evidence at 1% that the population mean exceeds 120.

PRACTICE Question 2

Measurements are normal with known $\sigma = 6$. An independent random sample of 16 gives $\bar{x} = 36.7$. Test at 5% whether the population mean differs from 40.

1. A difference requires two tails.

$$H_0 : \mu = 40, \quad H_1 : \mu \neq 40$$

2. Calculate the statistic.

$$SE = 6/\sqrt{16} = 1.5, \quad z = \frac{36.7 - 40}{1.5} = -2.2$$

3. Since $-2.2 < -1.96$, the result is in the critical region.

Answer. Reject H_0 . There is sufficient evidence at 5% that the population mean differs from 40.

PRACTICE Question 3

For a normal population with known $\sigma = 10$, take an independent random sample of 25. Express the 5% critical region for testing $H_0 : \mu = 100$ against $H_1 : \mu > 100$ in terms of the observed mean $\bar{x}$.

1. The standard error is two, and the test is upper-tailed.

$$\frac{\bar{x} - 100}{2} > 1.645$$

2. Rearrange the inequality.

$$\bar{x} > 100 + 2(1.645) = 103.29$$

Answer. Reject when $\bar{x} > 103.29$, using the rounded critical value.

PRACTICE Question 4

An independent random sample of 64 gives $\bar{x} = 19.4$ and $s = 4$. A large-sample normal approximation is suitable. Test at 5% whether the population mean is below 20.

1. State the hypotheses and estimate the standard error.

$$H_0 : \mu = 20, \quad H_1 : \mu < 20, \quad \widehat{SE} = 4/\sqrt{64} = 0.5$$

2. Compare the approximate statistic with the lower-tail threshold.

$$z_{\text{approx}} = \frac{19.4 - 20}{0.5} = -1.2 > -1.645$$

Answer. Do not reject H_0 . This approximate test provides insufficient evidence at 5% that the population mean is below 20.

Ready to move on?

- ☐ I can write hypotheses using the population mean.
- ☐ I can choose the correct tail or tails before calculating.
- ☐ I can state the conditions for the statistic I use.
- ☐ I can distinguish a known standard error from an estimated one.
- ☐ I can give a contextual conclusion without claiming to prove the null hypothesis.

Next: Confidence intervals foundations. Estimate a population mean with an interval that expresses sampling uncertainty.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

Cambridge 9709 syllabus, section 6.5: Hypothesis tests · Cambridge 9231 prior-learning guidance