

LESSON 08 9709 TO 9231

Poisson distribution

Model event counts in a fixed interval and turn model probabilities into expected frequencies.

By the end of this lesson

- Recognise the assumptions of a Poisson model.
- Match the parameter to the interval and calculate probabilities.
- Use Poisson moments and recover a parameter from a zero-count probability.
- Calculate expected frequencies for later goodness-of-fit work.

Before you begin: Discrete random variables and probability sums; Complementary probabilities; Factorials, exponentials and logarithms; Mean, variance and standard deviation.

1. What a Poisson variable counts

A Poisson variable counts events in a fixed interval of time, length, area or volume. Examples might include calls arriving in ten minutes or flaws along one metre of material. The interval is fixed; the event count is random. Unlike the binomial distribution, the model has no fixed maximum count.

KEY FORMULA Model and support

$$X \sim \text{Po}(\lambda), \quad X \in \{0, 1, 2, \dots\}$$

Take $\lambda > 0$. The parameter λ is the expected event count in the particular interval being modelled.

- The average event rate is constant over the interval.
- Counts in disjoint intervals are independent.
- Events do not systematically arrive in clusters or batches.
- The chance of two or more events in a sufficiently short interval is negligible compared with the chance of one.

Use the context to assess these assumptions. A rush-hour surge may contradict a constant rate; linked equipment failures may contradict independence. Exactly scheduled arrivals every ten minutes have a predictable pattern and do not fit the random-arrival model. A stated average rate alone is not enough to establish a Poisson distribution.

2. Probabilities of individual counts

KEY FORMULA Poisson probability

$$P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!}$$

For integer $r \geq 0$ when $X \sim \text{Po}(\lambda)$. The notation $e^{-\lambda}$ is also written $\exp(-\lambda)$.

The factorial applies to the observed count, not to the parameter. At zero, use $0! = 1$ and $\lambda^0 = 1$, giving $P(X = 0) = e^{-\lambda}$. Keep the full exponential value during calculation, especially if the probability will later be multiplied by a sample size.

WORKED EXAMPLE Worked example 1: exactly three faults

An illustrative model gives a mean of 2.4 faults per kilometre of cable. Assume faults follow a Poisson model. Find the probability of exactly three faults in one kilometre.

- Let X count faults in one kilometre, so $X \sim \text{Po}(2.4)$.
- Substitute the expected count and the required integer count.

$$P(X = 3) = \frac{e^{-2.4}(2.4)^3}{3!}$$

- Evaluate the power and factorial.

$$P(X = 3) = e^{-2.4} \frac{13.824}{6} = 0.2090141 \dots$$

Answer. The probability is approximately 0.209.

COMMON MISTAKE A mean can be non-integer

$\lambda = 2.4$ is an expected count, so it need not be an integer. The actual variable X still takes integer values only; $X = 2.4$ is not a possible count.

3. Match the parameter to the interval

Convert the rate before calculating any probability. If a process has constant mean rate a events per unit and the requested interval has length t in the same units, its expected count is at . A shorter interval gives a smaller Poisson parameter. Scale the expected count, not the finished probability.

KEY FORMULA Interval scaling

$$\lambda = \text{mean rate} \times \text{interval length}$$

The rate and length must use matching units, and the Poisson assumptions must remain reasonable throughout the interval.

WORKED EXAMPLE Worked example 2: at least one call

Calls follow a Poisson model with rate 6 per hour. Find the probability of at least one call in a 15-minute interval.

1. Convert 15 minutes to a fraction of an hour.

$$\lambda = 6 \left(\frac{15}{60} \right) = 1.5$$

2. Let $Y \sim \text{Po}(1.5)$. “At least one” is the complement of zero.

3. Use the zero-count probability.

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-1.5} = 0.7768698 \dots$$

Answer. The probability is approximately 0.777.

METHOD TIP Write the interval beside the parameter

Writing “ $\lambda = 1.5$ per 15-minute interval” makes the conversion visible. Using $\lambda = 6$ here would calculate an hourly probability instead.

4. Mean, variance and an unknown parameter**KEY FORMULA** Poisson moments

$$E(X) = \lambda, \quad \text{Var}(X) = \lambda, \quad \text{SD}(X) = \sqrt{\lambda}$$

For $X \sim \text{Po}(\lambda)$. The numerical values of the mean and variance are equal; the standard deviation is their square root.

For $X \sim \text{Po}(4)$, the mean is 4 events and the standard deviation is 2 events. Equality of mean and variance is a feature of the model, not a rule for every observed dataset. Sample summaries fluctuate; compare the model with the process assumptions as well as the data.

A zero-count probability can determine the parameter. If $P(X = 0) = a$, where $0 < a < 1$, then $e^{-\lambda} = a$. Taking natural logarithms gives $-\lambda = \ln a$, so $\lambda = -\ln a$. The negative sign is necessary because $\ln a$ is negative for a probability between 0 and 1.

KEY FORMULA Recovering the parameter from zero counts

$$\lambda = -\ln(P(X = 0))$$

Use when the zero-count probability is known under a Poisson model and lies strictly between 0 and 1.

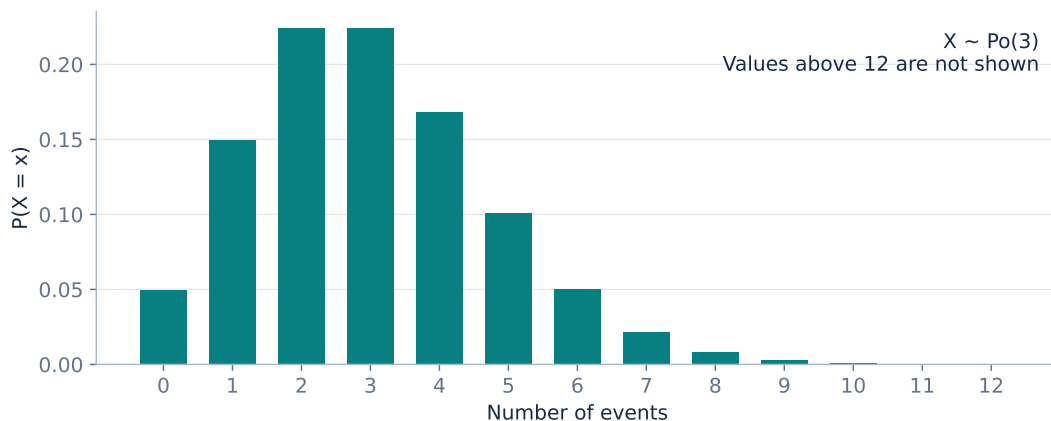
5. Several counts and complementary tails

To find a cumulative probability, add the individual probabilities for the allowed counts. For example, $P(X \leq 2)$ includes 0, 1 and 2. An upper tail is often shorter to calculate using a complement: $P(X \geq 3) = 1 - P(X \leq 2)$. Because the counts are integers, “more than 3” instead means $1 - P(X \leq 3)$.

KEY FORMULA A useful small cumulative sum

$$P(X \leq 2) = e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} \right)$$

For $X \sim \text{Po}(\lambda)$; the three terms correspond to counts 0, 1 and 2.



Probabilities for $X \sim \text{Po}(3)$, displayed for counts 0 to 12. Larger counts remain possible; the omitted probability above 12 is about 0.0000161.

The full collection of probabilities sums to 1. A truncated chart may omit a small positive tail. The largest bar does not have to identify a unique most likely value: when $\lambda = 3$, counts 2 and 3 have equal probabilities.

6. From model probabilities to expected frequencies

An expected frequency is the number predicted on average in a category across repeated observations. It differs from an observed frequency, which is an actual integer count. Later, a goodness-of-fit test compares observed frequencies with the frequencies predicted by a model.

KEY FORMULA Expected frequency

$$E_i = Np_i$$

N is the number of observations or intervals, and p_i is the model probability of category i . Categories should be disjoint and exhaustive.

WORKED EXAMPLE Worked example 3: predicted counts across intervals

The count in each of 120 independent intervals follows $\text{Po}(2)$. Find expected frequencies for counts 0, 1, 2 and 3 or more.

1. Calculate the four category probabilities.

$$p_0 = e^{-2}, \quad p_1 = 2e^{-2}, \quad p_2 = 2e^{-2}, \quad p_{3+} = 1 - 5e^{-2}$$

2. Multiply each probability by 120, retaining full precision.

$$E_0 = 16.2402\dots, \quad E_1 = 32.4805\dots, \quad E_2 = 32.4805\dots$$

3. Find the final category and check the total.

$$E_{3+} = 38.7988\dots, \quad \sum E_i = 120$$

Answer. The expected frequencies are approximately 16.240, 32.480, 32.480 and 38.799, respectively.

COMMON MISTAKE Do not round expected frequencies to whole counts

The values describe model averages and can be fractional. Keep their decimal precision for subsequent calculations; rounding each one to an integer changes the model comparison.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

$X \sim \text{Po}(4)$. Find (a) $P(X \leq 2)$ and (b) $P(X \geq 3)$.

1. Sum the probabilities for zero, one and two.

$$P(X \leq 2) = e^{-4} \left(1 + 4 + \frac{4^2}{2!} \right) = 13e^{-4} = 0.2381033\dots$$

2. The second event is the complement.

$$P(X \geq 3) = 1 - 13e^{-4} = 0.7618967\dots$$

Answer. (a) 0.238; (b) 0.762.

PRACTICE Question 2

A Poisson variable has $P(X = 0) = 0.2$. Find its parameter, mean and variance.

1. Equate the zero-count formula to the given probability.

$$e^{-\lambda} = 0.2$$

2. Take natural logarithms and change the sign.

$$\lambda = -\ln(0.2) = 1.6094379 \dots$$

3. Under a Poisson model, both $E(X)$ and $\text{Var}(X)$ equal λ .

Answer. The parameter, mean and variance are all approximately 1.61.

PRACTICE Question 3

Events follow a Poisson model at rate 12 per hour. For a 10-minute interval, find the model, the mean and standard deviation, and the probability of exactly one event.

1. Convert the interval to hours.

$$\lambda = 12 \left(\frac{10}{60} \right) = 2, \quad Y \sim \text{Po}(2)$$

2. Use the moment formulas.

$$E(Y) = 2, \quad \text{SD}(Y) = \sqrt{2} \approx 1.41$$

3. Apply the individual-count formula.

$$P(Y = 1) = 2e^{-2} = 0.2706706 \dots$$

Answer. $Y \sim \text{Po}(2)$; mean 2 events, standard deviation about 1.41 events, and $P(Y = 1) \approx 0.271$.

PRACTICE Question 4

Two hundred independent intervals each have a count $X \sim \text{Po}(3)$. Find (a) the expected number of zero-count intervals and (b) the probability of at least four events in one interval.

1. Multiply the zero-count probability by the number of intervals.

$$E_0 = 200e^{-3} = 9.9574137 \dots$$

2. Exclude counts zero through three for the upper tail.

$$P(X \geq 4) = 1 - e^{-3} \left(1 + 3 + \frac{9}{2} + \frac{27}{6} \right) = 0.3527681 \dots$$

Answer. (a) About 9.96 zero-count intervals on average; (b) approximately 0.353.

Ready to move on?

- ☐ I can state when a Poisson model is reasonable.
- ☐ I match the parameter to the requested interval before calculating.
- ☐ I can find individual, cumulative and complementary probabilities.
- ☐ I know the Poisson mean, variance and standard deviation.
- ☐ I can calculate expected frequencies and retain their precision.

Next: Linear combinations of random variables. Use distribution means and variances to study sums, differences and scaled measurements.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

[Cambridge 9709 syllabus, section 6.1: Poisson distribution](#) · [Cambridge 9231 prior-learning guidance](#)