

LESSON 06 9709 TO 9231

Continuous random variables

Use area and integration to find probabilities, averages and percentiles for measurements.

By the end of this lesson

- Check and normalise a probability density function.
- Calculate interval probabilities and percentiles.
- Find the mean, variance and standard deviation by integration.
- Handle a simple density with an infinite support.

Before you begin: Integrate powers of x using definite integrals; Rearrange equations and work with exact fractions; Recall expectation and variance for a discrete random variable.

1. Probability is an area

A continuous model describes a measurement that can take any value in an interval, such as time or length. Its probability density function, written $f(x)$, shows how probability is distributed. The support is the set of possible values in the model; the density is zero outside it.

KEY FORMULA Conditions for a valid density

$$f(x) \geq 0, \quad \int_{\text{whole support}} f(x) \, dx = 1$$

Check nonnegativity throughout the support as well as total area. Both conditions are necessary.

A density height is not a probability. For example, a rectangle of height 2 and width 0.5 has area 1, so it can define a valid density. If X is measured in seconds, its density has units of inverse seconds; multiplying height by width produces a dimensionless probability.

KEY FORMULA Probability between two values

$$P(a < X < b) = \int_a^b f(x) \, dx$$

Integrate only where the interval meets the support. For a continuous variable, $P(X = a) = 0$, so including a single endpoint does not change the answer.

COMMON MISTAKE Density is not point probability

Do not write $P(X = a) = f(a)$. A single point has zero width and therefore zero area, even where the density is positive.

2. Find the constant before the probability

If a density contains an unknown constant, use total probability 1 to find it. Then substitute the constant into the density before calculating other areas. Sketching the support first helps you choose sensible integration limits.

WORKED EXAMPLE A rising density

The density is $f(x) = kx$ for $0 < x < 2$ and zero elsewhere. Find k and $P(0.5 < X < 1.5)$.

1. Set the area over the entire support equal to one.

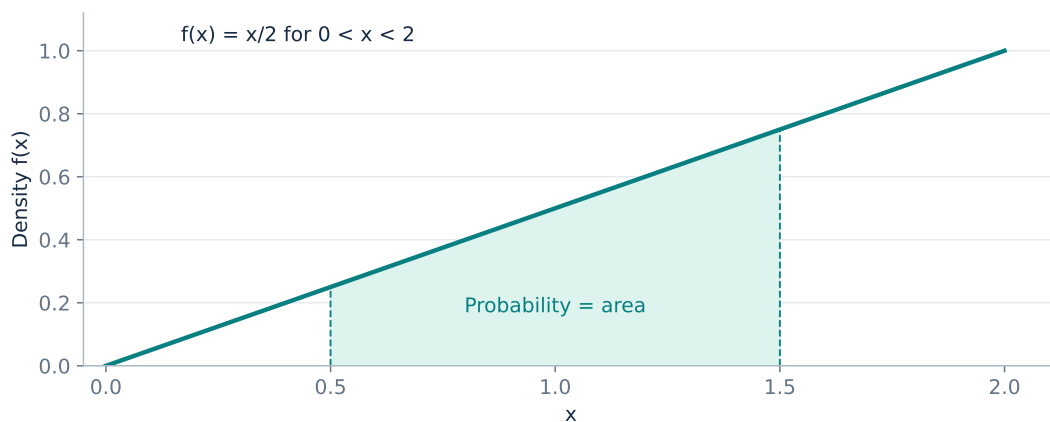
$$\int_0^2 kx \, dx = k \left[\frac{x^2}{2} \right]_0^2 = 2k = 1$$

2. Hence $k = \frac{1}{2}$. The resulting density is nonnegative throughout its support.

3. Integrate this density over the requested interval.

$$P(0.5 < X < 1.5) = \left[\frac{x^2}{4} \right]_{0.5}^{1.5} = \frac{2.25 - 0.25}{4} = \frac{1}{2}$$

Answer. The normalising constant is $k = \frac{1}{2}$ and the probability is 0.5.



For $f(x) = x/2$ on $0 < x < 2$, the shaded area between 0.5 and 1.5 is 0.5. The total triangular area is 1.

3. Mean and variance of a continuous model

Expectation uses the same weighting idea as a discrete distribution, with integration replacing summation. The mean is the long-run average of observations from the model. First calculate the two moments below, and then subtract the square of the mean.

KEY FORMULA Continuous moments

$$E(X) = \int x f(x) dx, \quad E(X^2) = \int x^2 f(x) dx$$

Both integrals are over the whole support, and the moments must exist.

KEY FORMULA Variance and standard deviation

$$\text{Var}(X) = E(X^2) - [E(X)]^2, \quad \text{SD}(X) = \sqrt{\text{Var}(X)}$$

Variance has squared measurement units. Standard deviation has the same units as X .

WORKED EXAMPLE Moments of the rising density

For $f(x) = x/2$ on $0 < x < 2$, find the mean, variance and standard deviation.

1. Multiply the density by x for the first moment.

$$E(X) = \int_0^2 \frac{x^2}{2} dx = \left[\frac{x^3}{6} \right]_0^2 = \frac{4}{3}$$

2. Multiply the density by x squared for the second moment.

$$E(X^2) = \int_0^2 \frac{x^3}{2} dx = \left[\frac{x^4}{8} \right]_0^2 = 2$$

3. Subtract using exact fractions.

$$\text{Var}(X) = 2 - \left(\frac{4}{3} \right)^2 = \frac{2}{9}, \quad \text{SD}(X) = \frac{\sqrt{2}}{3}$$

Answer. Mean = $\frac{4}{3}$, variance = $\frac{2}{9}$ and standard deviation ≈ 0.471 . The mean lies within the support and the variance is positive.

4. Medians and percentiles divide the area

The median m has half the probability to its left. More generally, a percentile is a value below which a specified proportion of the distribution lies. These are statements about area, so the median need not be halfway between the support endpoints.

KEY FORMULA Find a percentile

$$\int_{\text{lower support limit}}^q f(x) dx = p$$

For a median use $p = 0.5$; for the upper quartile use $p = 0.75$. Select a solution within the support.

For $f(x) = x/2$ on $0 < x < 2$, the area to the left of q is $q^2/4$. The median solves $m^2/4 = 1/2$, giving $m = \sqrt{2}$. The upper quartile solves $q^2/4 = 3/4$, giving $q = \sqrt{3}$. Both exceed 1, which fits the greater density towards the right.

5. When the support continues indefinitely

An infinite integration limit means a limit of finite integrals. Replace infinity by a number b , integrate, and then let b increase without bound. Total probability must converge to 1. A valid density does not automatically have a finite mean or variance: those integrals need checking separately.

WORKED EXAMPLE An infinite tail

Let $f(x) = 3/x^4$ for $x \geq 1$ and zero elsewhere. Check the density and find $P(X > 2)$, the mean and the variance.

1. The density is nonnegative for $x \geq 1$. Its total area is

$$\lim_{b \rightarrow \infty} \int_1^b 3x^{-4} dx = \lim_{b \rightarrow \infty} (1 - b^{-3}) = 1$$

2. For the tail probability, change the lower limit to two.

$$P(X > 2) = \lim_{b \rightarrow \infty} \left(\frac{1}{8} - b^{-3} \right) = \frac{1}{8}$$

3. Both moment integrals converge.

$$E(X) = \int_1^{\infty} 3x^{-3} dx = \frac{3}{2}, \quad E(X^2) = \int_1^{\infty} 3x^{-2} dx = 3$$

4. Use the variance identity.

$$\text{Var}(X) = 3 - \left(\frac{3}{2} \right)^2 = \frac{3}{4}$$

Answer. This is a valid density. Its tail probability is $\frac{1}{8}$, mean $\frac{3}{2}$ and variance $\frac{3}{4}$.

6. A reliable calculation routine

- Identify the support and set the density to zero elsewhere.
- Normalise if necessary, then select the integral that matches the question.
- Use exact fractions until the final numerical answer.
- Check probabilities lie between zero and one, percentiles lie in the support, and variances are non-negative.

METHOD TIP Check what you are integrating

For a probability integrate $f(x)$; for a mean integrate $xf(x)$; for a second moment integrate $x^2f(x)$. These are three different calculations.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

Let $f(x) = 2x$ for $0 < x < 1$, zero elsewhere. Find (a) $P(X > 0.5)$ and (b) the median.

1. Integrate from the threshold to the upper support limit.

$$P(X > 0.5) = \int_{0.5}^1 2x \, dx = [x^2]_{0.5}^1 = \frac{3}{4}$$

2. The median has area one half to its left.

$$\int_0^m 2x \, dx = m^2 = \frac{1}{2}$$

Answer. (a) $\frac{3}{4}$. (b) $m = 1/\sqrt{2} \approx 0.707$, taking the positive root within the support.

PRACTICE Question 2

A density is $f(x) = k(1+x)$ on $0 < x < 1$, zero elsewhere. Find k , the mean and the variance.

1. Normalise over the support.

$$k \int_0^1 (1+x) \, dx = \frac{3k}{2} = 1 \quad \Rightarrow \quad k = \frac{2}{3}$$

2. Calculate both moments.

$$E(X) = \frac{2}{3} \left(\frac{1}{2} + \frac{1}{3} \right) = \frac{5}{9}, \quad E(X^2) = \frac{2}{3} \left(\frac{1}{3} + \frac{1}{4} \right) = \frac{7}{18}$$

3. Subtract the squared mean.

$$\text{Var}(X) = \frac{7}{18} - \frac{25}{81} = \frac{13}{162}$$

Answer. $k = \frac{2}{3}$, mean = $\frac{5}{9}$ and variance = $\frac{13}{162}$.

PRACTICE Question 3

A density equals 4 on $1 < x < 1.25$ and zero elsewhere. Explain why it is valid, then find $P(1.1 < X < 1.2)$ and the median.

1. The density is nonnegative and its total area is $4(1.25 - 1) = 1$.

2. The requested probability is the rectangle area $4(1.2 - 1.1) = 0.4$.

3. For the median, $4(m - 1) = 0.5$, so $m = 1.125$.

Answer. The density is valid despite its height exceeding 1. The probability is 0.4 and the median is 1.125.

PRACTICE Question 4

Let $f(x) = 3x^2$ for $0 < x < 1$, zero elsewhere. Find the mean, variance and upper quartile.

1. Calculate the moments over the support.

$$E(X) = \int_0^1 3x^3 \, dx = \frac{3}{4}, \quad E(X^2) = \int_0^1 3x^4 \, dx = \frac{3}{5}$$

2. Find the variance.

$$\text{Var}(X) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

3. Set the area to the left of the upper quartile equal to 0.75.

$$\int_0^q 3x^2 \, dx = q^3 = \frac{3}{4}$$

Answer. Mean = $\frac{3}{4}$, variance = $\frac{3}{80}$ and upper quartile $q = (3/4)^{1/3} \approx 0.909$.

Ready to move on?

- ☐ I can distinguish a density height from a probability.
- ☐ I can use the support to choose correct integration limits.
- ☐ I can calculate moments and subtract the squared mean.
- ☐ I can find a percentile by solving an area equation.
- ☐ I can evaluate a simple improper integral using a limit.

Next: The normal distribution. Apply the same area interpretation to an important continuous model, then use standardisation to calculate its probabilities.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

Cambridge 9709 syllabus, section 6.3: Continuous random variables · Cambridge 9231 prior-learning guidance