

LESSON 05 9709 TO 9231

Geometric distribution

Calculate how long a sequence of independent trials waits for its first success.

By the end of this lesson

- Recognise a geometric experiment and define its trial count.
- Calculate first-success and tail probabilities.
- Use the mean to interpret or determine the success probability.
- Distinguish a waiting-time variable from a binomial success count.

Before you begin: Independent trials and complementary probabilities; Powers and simple probability products; The meaning of an expected value.

1. Count trials up to and including success

A geometric variable counts how many trials are needed to obtain the first success. The stopping point is random: the first trial may succeed, or many failures may occur first. Use the same definition of success on every trial. As with a binomial model, success can mean any outcome being studied, including a fault.

- Trials are independent.
- Every trial has success or failure as its two possible categories.
- The success probability p is constant, with $0 < p \leq 1$.
- Continue until the first success, including that successful trial in the count.

KEY FORMULA Model and support

$$X \sim \text{Geo}(p), \quad X \in \{1, 2, 3, \dots\}$$

Cambridge counts trials including the first success. Write $q = 1 - p$ for the failure probability.

The smallest possible value is 1, because success on the first trial still uses one trial. There is no fixed largest possible value when $0 < p < 1$. Some sources define a geometric variable as the number of failures before success; that variable starts at 0. Always state your definition to avoid shifting every answer by one.

2. First success on a specified trial

The event $X = r$ requires exactly $r - 1$ failures, followed by one success. This ordering is compulsory: an earlier success would stop the experiment. Independence allows the probabilities to be multiplied. No combinations factor is needed because there is only one permitted success-failure pattern.

KEY FORMULA Geometric probability

$$P(X = r) = (1 - p)^{r-1}p = q^{r-1}p$$

For integer $r \geq 1$, with independent trials and constant success probability $0 < p \leq 1$.

WORKED EXAMPLE Worked example 1: first success on trial 4

An independent attempt succeeds with probability 0.3. Attempts continue until success. Find the probability that the first success occurs on attempt 4.

- Let X count attempts including the successful attempt. Then $X \sim \text{Geo}(0.3)$.
- Trial 4 must be preceded by three failures: failure, failure, failure, success.

$$P(X = 4) = (0.7)^3(0.3)$$

- Evaluate the product.

$$P(X = 4) = 0.1029$$

Answer. The probability is 0.1029, approximately 0.103.

COMMON MISTAKE The exponent counts failures

For first success on trial 4, use three failures, not four. The expression $(0.7)^4(0.3)$ would describe first success on trial 5.

3. After, at least and within

Tail probabilities have a particularly simple form. If $X > r$, the first r trials must all fail. There is no need to specify when success happens after them. If $X \geq r$, only the first $r - 1$ trials must fail: success on trial r is allowed.

KEY FORMULA After r trials

$$P(X > r) = q^r, \quad P(X \leq r) = 1 - q^r$$

Use integer $r \geq 0$. "Within the first r trials" means $X \leq r$.

KEY FORMULA At least r trials

$$P(X \geq r) = q^{r-1}$$

Use integer $r \geq 1$. Compare the included starting value carefully.

WORKED EXAMPLE Worked example 2: success within five attempts

Each independent attempt succeeds with probability 0.2. Find the probability of (a) needing more than five attempts and (b) succeeding within five attempts.

1. Let $X \sim \text{Geo}(0.2)$, so the failure probability is $q = 0.8$.

2. More than five attempts requires five initial failures.

$$P(X > 5) = (0.8)^5 = 0.32768$$

3. Within five attempts is the complementary event.

$$P(X \leq 5) = 1 - (0.8)^5 = 0.67232$$

Answer. (a) 0.32768; (b) 0.67232. They add to 1.

For an interval, subtract tails. First success from trial a through trial b , inclusive, means waiting at least a trials but not more than b . Thus $P(a \leq X \leq b) = q^{a-1} - q^b$ for integers $1 \leq a \leq b$. Listing the endpoint trials before calculating helps prevent an off-by-one error.

4. Expected waiting time

The expected value describes the average number of trials over many repetitions of the whole waiting experiment. A smaller success probability gives a longer expected wait. This average is not a deadline, and success does not become more likely merely because several failures have occurred.

KEY FORMULA Mean number of trials

$$E(X) = \frac{1}{p}$$

For $X \sim \text{Geo}(p)$ with $0 < p \leq 1$, counting the successful trial.

WORKED EXAMPLE Worked example 3: recovering the probability

A repeated independent experiment has a geometric waiting time with mean 5 trials. Find the success probability and the probability of success within the first three trials.

1. Use the expected-value formula to recover the probability.

$$\frac{1}{p} = 5 \implies p = 0.2$$

2. The failure probability is 0.8. Within three trials is the complement of three failures.

$$P(X \leq 3) = 1 - (0.8)^3 = 1 - 0.512$$

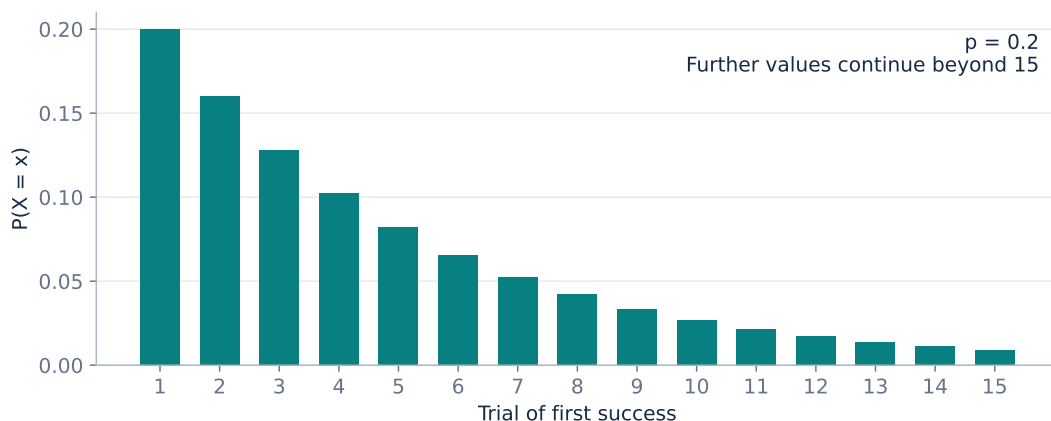
3. Evaluate and interpret the cumulative probability.

$$P(X \leq 3) = 0.488$$

Answer. The success probability per trial is 0.2, and the chance of success within three trials is 0.488.

METHOD TIP Check the scale

An expected waiting time of 5 trials does not mean that trial 5 is the most likely first-success trial. For $0 < p < 1$, trial 1 always has the largest individual first-success probability.

5. Shape and the effect of previous failures

First-success probabilities for $X \sim \text{Geo}(0.2)$, shown for trials 1 to 15. The omitted tail is

$$P(X > 15) = 0.8^{15} \approx 0.0352.$$

For $0 < p < 1$, each bar is q times the preceding bar, so probabilities decrease as the trial number increases. Nevertheless, many later outcomes together can have a substantial total probability. The visible bars need not sum to 1 when the graph stops at a finite trial number.

Independence also explains why past failures do not alter the next attempt: after ten failures, the next trial still succeeds with probability p . More generally, given r failures, the probability of waiting more than

another s trials is q^s . This property is called memorylessness; it follows from the same unchanged trial probabilities.

6. Choosing the right trial model

Feature	Binomial	Geometric
What is counted?	Successes in a fixed number of trials	Trials until first success
What is fixed?	Number of trials n	Success probability p ; stop at success
Possible values	$0, 1, \dots, n$	$1, 2, 3, \dots$
Mean	np	$1/p$

Both models need independent trials with constant success probability. First decide what the random variable counts; similar words in the story can lead to different models. If success becomes easier with practice, or each failure changes the next attempt, the constant-probability assumption needs reconsidering.

COMMON MISTAKE Do not attach an extra success factor to a tail

$P(X > r)$ only specifies initial failures, so it is q^r . Multiplying by p would instead require success immediately after those failures.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

$X \sim \text{Geo}(0.4)$. Find (a) $P(X = 3)$, (b) $P(X > 3)$ and (c) $P(X \geq 3)$.

1. First success on trial 3 needs two failures then success.

$$P(X = 3) = (0.6)^2(0.4) = 0.144$$

2. More than three means three failures; at least three means two failures.

$$P(X > 3) = (0.6)^3 = 0.216, \quad P(X \geq 3) = (0.6)^2 = 0.36$$

Answer. (a) 0.144; (b) 0.216; (c) 0.360.

PRACTICE Question 2

$Y \sim \text{Geo}(0.2)$. Find the probability that the first success is on trial 3, 4, 5 or 6.

1. The required event includes both endpoints: $3 \leq Y \leq 6$. Subtract the tail beyond trial 6 from the tail starting at trial 3.

$$P(3 \leq Y \leq 6) = P(Y \geq 3) - P(Y > 6)$$

2. Use the failure probability 0.8.

$$P(3 \leq Y \leq 6) = (0.8)^2 - (0.8)^6 = 0.377856$$

Answer. The probability is approximately 0.378.

PRACTICE Question 3

A geometric waiting time T has mean 4 trials. Find its success probability, $P(T = 4)$ and $P(T > 4)$.

1. Invert the mean to find the probability.

$$p = \frac{1}{4} = 0.25$$

2. Use three failures then success for the exact count.

$$P(T = 4) = (0.75)^3(0.25) = 0.10546875$$

3. Use four failures for the tail.

$$P(T > 4) = (0.75)^4 = 0.31640625$$

Answer. $p = 0.25$; $P(T = 4) \approx 0.105$; $P(T > 4) \approx 0.316$.

PRACTICE Question 4

Trials are independent with success probability 0.3. In experiment A, six trials are performed and successes counted. In experiment B, trials continue until the first success. State both models and find the chance that experiment B finishes within four trials.

1. Experiment A counts successes in fixed trials, so $A \sim B(6, 0.3)$. Experiment B counts waiting trials, so $B \sim \text{Geo}(0.3)$.

2. Finishing within four trials means at least one success in those trials.

$$P(B \leq 4) = 1 - (0.7)^4 = 0.7599$$

Answer. $A \sim B(6, 0.3)$, $B \sim \text{Geo}(0.3)$; the required probability is 0.7599.

Ready to move on?

- ☐ I count the successful trial and start the support at 1.
- ☐ I can explain why there is no combinations factor.
- ☐ I can distinguish first success on, after, and by a specified trial.
- ☐ I can use the geometric mean and interpret it correctly.
- ☐ I can choose between binomial and geometric variables.

Next: Continuous random variables. Move from probabilities at separate integer values to probabilities represented by areas under a density curve.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

[Cambridge 9709 syllabus, section 5.4: discrete random variables](#) · [Cambridge 9231 prior-learning guidance](#)