

LESSON 02 9709 TO 9231

Probability and combinations

Translate chance statements into events and calculate probabilities for simple selections and repeated trials.

By the end of this lesson

- Use complements, unions, intersections and conditional probabilities.
- Distinguish mutually exclusive events from independent events.
- Calculate with probability trees and combinations.

Before you begin: Fractions, decimals and percentages; Basic set notation is helpful but introduced here.

1. Outcomes, events and complements

A random experiment has an uncertain outcome, such as the result of rolling a die. Its sample space is the set of all possible outcomes. An event is a subset of that space. For a fair six-sided die, the event $A = \text{"an even result"}$ is $\{2, 4, 6\}$.

Notation	Meaning
$A \cap B$	Both A and B occur
$A \cup B$	A or B or both occur
A'	A does not occur

KEY FORMULA Equally likely outcomes

$$P(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in the sample space}}$$

Use counting only when all outcomes in the chosen sample space are equally likely.

KEY FORMULA The complement rule

$$P(A') = 1 - P(A)$$

An event and its complement cover every outcome and cannot occur together.

Every probability lies between 0 and 1. Impossible events have probability 0 and certain events have probability 1. A complement often simplifies an “at least one” question: no successes may be easier to calculate than one, two or more successes separately.

2. Adding probabilities without double counting

Adding $P(A)$ and $P(B)$ counts the overlap twice. Subtract its probability once to obtain the probability of the union. The word “or” in this rule includes outcomes where both events occur.

KEY FORMULA General addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule applies whether or not the events are independent.

For a fair die, let A = “even” and B = “greater than 3”. Their intersection is $\{4, 6\}$ and their union is $\{2, 4, 5, 6\}$. Therefore $P(A \cup B) = 3/6 + 3/6 - 2/6 = 2/3$.

Mutually exclusive events cannot occur together, so $P(A \cap B) = 0$. Only in that case does the addition rule reduce to $P(A \cup B) = P(A) + P(B)$. Rolling a 1 and rolling a 6 on one die are mutually exclusive.

3. Conditional probability changes the reference group

$P(A | B)$ means the probability of A given that B has occurred. Once B is known, only outcomes inside B remain relevant. The part of this smaller reference group that also belongs to A is the intersection.

KEY FORMULA Conditional probability

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

The condition requires $P(B) > 0$. Reversing A and B usually changes the answer.

WORKED EXAMPLE Music and sport

Among 60 students, 24 study music, 30 play a sport and 18 do both. A student is chosen uniformly at random. Find the probability that they do either activity, then find the probability that they study music given that they play a sport.

- Count the union, removing the duplicated overlap.

$$P(M \cup S) = \frac{24 + 30 - 18}{60} = \frac{36}{60} = 0.6$$

- Given sport, the reference group is the 30 students who play a sport.

$$P(M | S) = \frac{18}{30} = 0.6$$

- For comparison, reversing the condition produces a different denominator.

$$P(S | M) = \frac{18}{24} = 0.75$$

Answer. The probability of either activity is 0.6. The conditional probability of music given sport is also 0.6, while the reverse conditional probability is 0.75.

METHOD TIP Read the condition first

In $P(A \mid B)$, identify the B group before choosing a denominator. A two-way table or a labelled Venn diagram can help organise the counts.

4. Repeated events and probability trees

A tree records successive choices. Probabilities leaving any one branch point sum to 1. Multiply along a complete path because later branch probabilities are conditional on the earlier outcomes. Add the probabilities of separate complete paths that lead to the requested event.

KEY FORMULA Multiplication along a path

$$P(A \cap B) = P(A)P(B \mid A)$$

This form requires $P(A) > 0$. Multiplying the two unconditional probabilities is justified when A and B are independent.

WORKED EXAMPLE Drawing without replacement

A bag contains 3 red and 2 blue tokens. Two tokens are drawn at random without replacement. Find the probability of different colours.

1. Different colours can occur through either path RB or BR.

$$P(RB) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$$

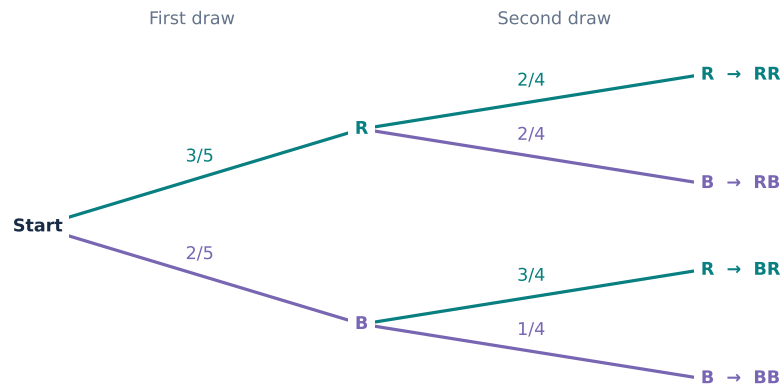
2. After a blue token is drawn, three red tokens remain among four tokens.

$$P(BR) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$$

3. These complete paths cannot both occur, so add them.

$$P(\text{different}) = \frac{3}{10} + \frac{3}{10} = \frac{3}{5}$$

Answer. The probability of different colours is $3/5 = 0.6$.



Two draws without replacement from 3 red and 2 blue tokens. The second-draw probabilities depend on the first colour.

COMMON MISTAKE Update the bag

Without replacement, the second denominator is 4. Its numerator also changes when the first token has the colour being counted.

5. Independence and mutual exclusivity

Two events are independent when learning that one has occurred does not change the probability of the other. If $P(B) > 0$, this means $P(A | B) = P(A)$. A useful test that also handles events of probability zero is the product rule below.

KEY FORMULA Independence test

$$P(A \cap B) = P(A)P(B)$$

This equality characterises independence of two events.

In the student survey, $P(M) = 24/60 = 0.4$ but $P(M | S) = 0.6$, so the events are dependent. The calculation identifies an association in this group; it does not establish that one activity causes the other.

Two mutually exclusive events with positive probabilities are dependent. If one occurs, the other becomes impossible. Their intersection probability is 0, while the product of their positive probabilities is greater than 0.

Replacing a token restores the original composition. With fresh random draws from the restored bag, the draw colours are independent, so $P(RR) = (3/5)^2 = 9/25$. Replacement alone should not replace a check that the drawing process is random.

6. Counting selections with combinations

The factorial $n!$ is the product $n(n-1) \cdots 2 \times 1$, with $0! = 1$. A combination counts an unordered selection of distinct objects. Selecting A and B produces the same pair as selecting B and A.

KEY FORMULA Selecting r objects from n

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For integers $0 \leq r \leq n$, selecting without replacement from n distinct objects. Order does not matter.

WORKED EXAMPLE Choosing a pair

A group contains 5 boys and 4 girls. Every pair of students is equally likely to be selected. Find the probability that the pair contains one boy and one girl.

- Count all unordered pairs from the nine students.

$$\binom{9}{2} = 36$$

- Choose one student from each group.

$$\binom{5}{1} \binom{4}{1} = 5 \times 4 = 20$$

- Divide the favourable count by the total count.

$$P(\text{one of each}) = \frac{20}{36} = \frac{5}{9}$$

Answer. The probability is $5/9$. Each pair is counted once in both numerator and denominator.

METHOD TIP Use one counting convention

Keep selections unordered throughout a combination calculation. Do not compare an ordered favourable count with an unordered total count.

Check your understanding

Try each question before reading its worked solution. Keep unrounded values during calculations.

PRACTICE Question 1

$P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cap B) = 0.25$. Find $P(A \cup B)$ and $P(A | B)$. Are the events independent?

- Remove the overlap when adding.

$$P(A \cup B) = 0.6 + 0.4 - 0.25 = 0.75$$

- Divide by the probability of the condition.

$$P(A | B) = \frac{0.25}{0.4} = 0.625$$

3. Compare the intersection with the product.

$$P(A)P(B) = 0.24 \neq 0.25$$

Answer. The probabilities are 0.75 and 0.625. The events are dependent.

PRACTICE Question 2

Two tokens are drawn without replacement from 4 green and 3 yellow tokens. Find the probabilities of (a) two green tokens and (b) different colours.

1. For two green tokens, update both numerator and denominator after the first draw.

$$P(GG) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

2. Add the two different-colour paths.

$$P(GY) + P(YG) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{4}{7}$$

Answer. (a) $2/7$. (b) $4/7$.

PRACTICE Question 3

A group of seven students contains four who study chemistry and three who do not. Three students are selected uniformly at random. Find the probability that exactly two study chemistry.

1. Count all possible groups of three.

$$\binom{7}{3} = 35$$

2. Choose two from the chemistry group and one from the other group.

$$\binom{4}{2} \binom{3}{1} = 6 \times 3 = 18$$

3. Use the ratio of the two counts.

$$P(\text{exactly two}) = \frac{18}{35}$$

Answer. The probability is $18/35 \approx 0.514$.

PRACTICE Question 4

Three independent trials each have success probability 0.3. Find the probability of at least one success.

1. The complementary event is no successes. Each failure has probability 0.7.

$$P(\text{no successes}) = 0.7^3 = 0.343$$

2. Subtract from one.

$$P(\text{at least one}) = 1 - 0.343 = 0.657$$

Answer. The probability is 0.657. Independence justifies multiplying the three failure probabilities.

Ready to move on?

- ☐ I can translate “and”, “or” and “not” into event notation.
- ☐ I can choose the correct denominator for a conditional probability.
- ☐ I can update probabilities after a draw without replacement.
- ☐ I can explain why independence and mutual exclusivity are different.
- ☐ I can count unordered selections and use a complement efficiently.

Next: Discrete random variables. A probability distribution assigns probabilities to numerical outcomes so that we can calculate a theoretical mean and variance.

Course alignment: selected 9709 Statistics 1 and 2 foundations for Further Mathematics 9231. Numerical examples are original teaching examples.

Cambridge 9709 syllabus, sections 5.2-5.3 · Cambridge 9231 prior-learning guidance