

Algebra: structure and complete proofs

LESSON 1

Symmetric expressions

C1.1 Let $x \neq 0$ and $x + \frac{1}{x} = 3$. Find $x^2 + \frac{1}{x^2}$.

C1.2 Let $x \neq 0$ and $x + \frac{1}{x} = 3$. Find $x^3 + \frac{1}{x^3}$.

C1.3 Let $a + b + c = 0$. Prove that $a^2 + b^2 + c^2 = -2(ab + bc + ca)$ and $a^3 + b^3 + c^3 = 3abc$.

C1.4 Real numbers a, b, c satisfy $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$. Find $a^4 + b^4 + c^4$.

C1.5 Let $x + y + z = 0$ and $x^2 + y^2 + z^2 = 2$. Prove that the numbers $x^3 - x$, $y^3 - y$, $z^3 - z$ are equal.

LESSON 2

Factorisation and primality

C2.1 Factor $8a^3 + 27b^3$.

C2.2 Factor $x^6 - 1$ into factors with integer coefficients.

C2.3 Factor $x^4 + 4y^4$.

C2.4 Find all positive integer pairs x, y for which $x^4 + 4y^4$ is prime.

C2.5 Find every positive integer n for which $n^5 + n + 1$ is prime.

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LESSON 3

Equations: make the search complete

C3.1 Solve in real numbers:

$$(x^2 - 3x)^2 - 2(x^2 - 3x) - 8 = 0.$$

C3.2 Find all real pairs (x, y) satisfying

$$x^2 + y = 12, \quad y^2 + x = 12.$$

C3.3 Find all ordered pairs of positive integers satisfying

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}.$$

C3.4 Find all positive integer pairs (a, b) satisfying

$$a^3 + b^3 = (a + b)^2.$$

C3.5 Positive integers a, b, c satisfy $3c^2 = c(a + b) + ab$, and $a - b$ is prime. Prove that $8c + 1$ is a perfect square.

LESSON 4

Inequalities from squares

C4.1 For $x > 0$, find the least value of $x + \frac{1}{x}$.

C4.2 Let $x, y > 0$. Prove that $\frac{x^2}{y} + \frac{y^2}{x} \geq x + y$.

C4.3 For $a, b > 0$, prove that $\frac{x^2}{a} + \frac{y^2}{b} \geq \frac{(x + y)^2}{a + b}$ for all real x, y .

C4.4 Positive real numbers x, y satisfy $x + y = 1$. Prove that

$$\left(\frac{1}{x^2} - 1\right) \left(\frac{1}{y^2} - 1\right) \geq 9.$$

C4.5 Let $a, b, c > 0$. Prove

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}.$$