CAMBRIDGE INTERNATIONAL AS & A LEVEL MATHEMATICS 9709

Cambridge 9709 Paper 3 Revision Booklet

Exam-Focused Practice and Summary Notes

Contents

How to Use This Booklet

Read each summary quickly, then attempt the questions independently. Use the solutions only after a proper attempt.

Mark weak methods immediately: algebraic manipulation, domain restrictions, trig intervals, integration choices, vector setup and differential-equation modelling.

Finish with the mixed Paper 3 practice section under timed conditions.

TOPIC REVISION

Trigonometry

Paper 3 Skills

- The cosecant, secant and cotangent ratios
- Compound angle formulae
- Double angle formulae
- Further trigonometric identities
- Expressing $a\sin\theta + b\cos\theta$ in R-form

Key Results

- $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$, $\cot x = \frac{1}{\tan x}$.
- $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$.
- $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$, and $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$.
- $a\sin x + b\cos x = R\sin(x+\alpha)$, where $R = \sqrt{a^2+b^2}$. Match coefficients carefully.

Common Traps

- Dividing by a trig expression without checking whether it could be zero.
- Giving only the calculator principal value instead of every solution in the interval.
- Using degree mode when the question is in radians.

Selected Practice

Question 1

Core

Nov 2021 p32 q1 · Source ID 9

Solve the equationrn
$\sec\theta = 3\cos\theta + 1$ for $0^\circ \le \theta \le 360^\circ$.

Question 2

Core

Feb/Mar 2022 p32 q5 · Source ID 151

The angles α and β are between 0° and 180° and satisfy the conditions:

$\tan(\alpha+\beta) = 2$ and $\tan\alpha = 3\tan\beta$.

Find the possible values of α and β.

Question 3

Core

Nov 2021 p32 q8 · Source ID 59

(a) By first expanding $(\cos^2\theta + \sin^2\theta)^2$, show that $\cos^4\theta + \sin^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$.

(b) Hence solve the equation $\cos^4\theta + \sin^4\theta = \frac{5}{9}$, for $0^\circ < \theta < 180^\circ$.

Question 4

Core

Nov 2023 p31 q1 · Source ID 63

Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$.

Question 5

Core

June 2023 P33 q6 · Source ID 156

(a) Express $3\cos x + 2\cos(x - 60°)$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. State the exact value of R and give α correct to 2 decimal places.

(b) Hence solve the equation $3\cos 2\theta + 2\cos(2\theta - 60°) = 2.5$ for $0° < \theta < 180°$.

TOPIC REVISION

Algebra

Paper 3 Skills

- The modulus function and modulus inequalities
- Division of polynomials, factor theorem and remainder theorem

Key Results

- $|x-a|<b$ means $a-b<x<a+b$, for $b>0$.
- $|f(x)|>k$ usually splits into $f(x)>k$ or $f(x)<-k$.
- If $p(a)=0$, then $(x-a)$ is a factor of $p(x)$.
- The remainder on division by $(ax+b)$ is $p(-b/a)$.

Common Traps

- Solving modulus inequalities as though the modulus signs were brackets.
- Using the wrong root for a linear divisor such as $2x-1$.

Selected Practice

Question 6

Core

June 2022 p33 q1 · Source ID 1489

Find, in terms of a, the set of values of x satisfying the inequality:

$2|3x + a| < |2x + 3a|$, where a is a positive constant.

Question 7

Exam

Nov 2023 p32 q1 · Source ID 1450

Diagram / sketch space

Use the original exam diagram if available.

(a) Sketch the graph of $y=|4x-2|$.

(b) Solve the inequality $1+3x<|4x-2|$.

Question 8

Core

Nov 2023 p33 q3 · Source ID 1412

The polynomial $2x^3 + ax^2 + bx + 6$, where a and b are constants, is denoted by $p(x)$. When $p(x)$ is divided by $(x + 2)$ the remainder is -38 and when $p(x)$ is divided by $(2x - 1)$ the remainder is $\frac{19}{2}$.

Find the values of a and b.

Question 9

Exam

Nov 2023 p32 q3 · Source ID 1423

The polynomial $2x^3 + ax^2 - 11x + b$ is denoted by $p(x)$. It is given that $p(x)$ is divisible by $(2x - 1)$ and that when $p(x)$ is divided by $(x + 1)$ the remainder is 12.

Find the values of a and b.

TOPIC REVISION

Logarithms and Exponentials

Paper 3 Skills

- Solving logarithmic equations
- Solving exponential equations
- Solving exponential inequalities
- Natural logarithms
- Transforming a relationship to linear form

Key Results

- $\log a + \log b = \log(ab)$, $\log a - \log b = \log(a/b)$, $k \log a = \log(a^k)$.
- For $\ln f(x)$, require $f(x) > 0$. Reject roots outside the log domain.
- For $y = ab^x$, use $\ln y = \ln a + x \ln b$.
- For $y = Ax^n$, use $\ln y = \ln A + n \ln x$.

Common Traps

- Keeping extraneous roots after combining logarithms.
- Forgetting that exponential expressions are always positive.
- Reading gradient/intercept from a transformed graph in the wrong variables.

Selected Practice

Question 10

Core

June 2022 p33 q3 · Source ID 1506

(a) Show that the equation $\log_3(2x+1) = 1 + 2\log_3(x-1)$ can be written as a quadratic equation in x.

(b) Hence solve the equation $\log_3(4y+1) = 1 + 2\log_3(2y-1)$, giving your answer correct to 2 decimal places.

Question 11

Core

Nov 2021 p33 q3 · Source ID 1493

Solve the equation $4^{x-2} = 4^x - 4^2$, giving your answer correct to 3 decimal places.

Question 12

Core

Nov 2023 p33 q1 · Source ID 1513

Find the set of values of x satisfying the inequality $|2^{x+1} - 2| < 0.5$, giving your answer to 3 significant figures.

Question 13

Core

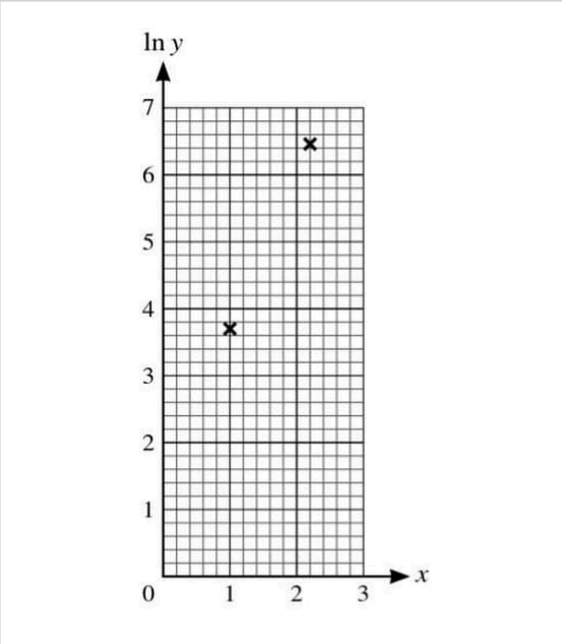
Nov 2022 p33 q1 · Source ID 1549

Solve the equation $\ln(2x-1) = 2\ln(x+1) - \ln x$. Give your answer correct to 3 decimal places.

Question 14

Core

Nov 2023 p31 q3 · Source ID 1570



The variables x and y are related by the equation $y = ab^x$, where a and b are constants. The diagram shows the result of plotting $\ln y$ against x for two pairs of values of x and y. The coordinates of these points are (1, 3.7) and (2.2, 6.46).

Use this information to find the values of a and b.

TOPIC REVISION

Differentiation

Paper 3 Skills

- Implicit differentiation
- Parametric differentiation
- Product, quotient, exponential, logarithmic and trigonometric derivatives

Key Results

- Product rule: $(uv)' = u'v + uv'$. Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$.
- $\frac{d}{dx}e^{kx} = ke^{kx}$, $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$.
- For parametric curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.
- Implicit differentiation: differentiate every term with y using $\frac{dy}{dx}$.

Common Traps

- Forgetting the chain rule inside exponentials, logs and trig functions.
- Using $\frac{dx}{dt} / \frac{dy}{dt}$ instead of $\frac{dy}{dt} / \frac{dx}{dt}$.

Selected Practice

Question 15

Core

Nov 2023 p33 q7 · Source ID 1646

The equation of a curve is $x^3 + y^2 + 3x^2 + 3y = 4$.

(a) Show that $\frac{dy}{dx} = -\frac{3x^2+6x}{2y+3}$.

(b) Hence find the coordinates of the points on the curve at which the tangent is parallel to the x-axis.

Question 16

Exam

June 2023 p32 q7 · Source ID 1657

The equation of a curve is $3x^2 + 4xy + 3y^2 = 5$.

(a) Show that $\frac{dy}{dx} = -\frac{3x+2y}{2x+3y}$.

(b) Hence find the exact coordinates of the two points on the curve at which the tangent is parallel to $y + 2x = 0$.

Question 17

Core

Nov 2023 p32 q2 · Source ID 1614

The parametric equations of a curve are

$x = (\ln t)^2$, $y = e^{2-t^2}$,

for $t > 0$.

Find the gradient of the curve at the point where $t = e$, simplifying your answer.

Question 18

Core

Nov 2023 p31 q1 · Source ID 1682

Find the exact coordinates of the points on the curve $y = \frac{x^2}{1-3x}$ at which the gradient of the tangent is equal to 8.

Question 19

Exam

Nov 2022 p32 q3 · Source ID 1594

The equation of a curve is $y = \sin x \sin 2x$. The curve has a stationary point in the interval $0 < x < \frac{1}{2}\pi$.

Find the x-coordinate of this point, giving your answer correct to 3 significant figures.

TOPIC REVISION

Integration

Paper 3 Skills

- Integration of $\frac{1}{ax+b}$ and algebraic fractions
- Integration by substitution
- Trigonometric integration
- Further integration and applications
- Partial fractions in integration

Key Results

- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$.
- $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$.
- $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$.
- Substitution must change limits in definite integrals, or substitute back before evaluating.
- Integration by parts: $\int u\,dv = uv - \int v\,du$.

Common Traps

- Dropping the modulus in logarithmic integrals when needed.
- Forgetting to square y in volumes of revolution.
- Changing variable but keeping old limits.

Selected Practice

Question 20

Core

Nov 2021 p32 q6 · Source ID 1721

(a) Using the expansions of $\sin(3x+2x)$ and $\sin(3x-2x)$, show that $\frac{1}{2}(\sin 5x + \sin x) \equiv \sin 3x \cos 2x$.

(b) Hence show that $\int_0^{\frac{1}{4}\pi} \sin 3x \cos 2x\,dx = \frac{1}{5}(3-\sqrt{2})$.

Question 21

Core

June 2022 p31 q6 · Source ID 1692

Let $I = \int_0^3 \frac{27}{(9+x^2)^2}\,dx$.

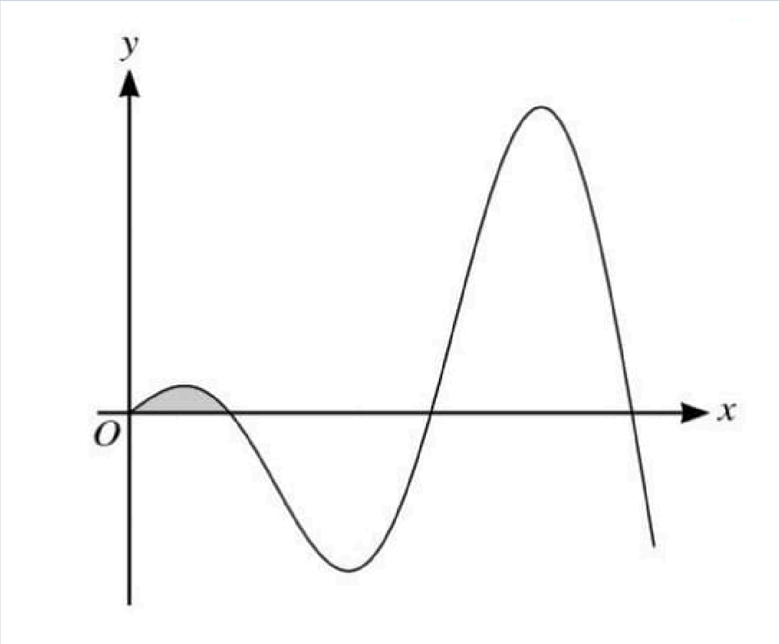
(a) Using the substitution $x = 3\tan\theta$, show that $I = \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$.

(b) Hence find the exact value of I.

Question 22

Core

Nov 2023 p33 q10 · Source ID 1755



The diagram shows the curve $y = x\cos 2x$, for $x \geq 0$.

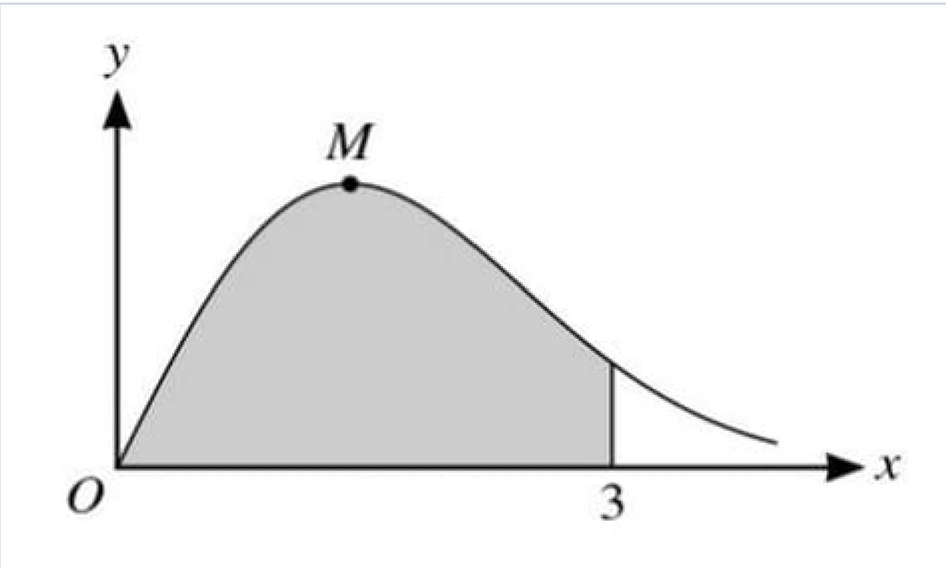
(a) Find the equation of the tangent to the curve at the point where $x = \frac{1}{2}\pi$.

(b) Find the exact area of the shaded region shown in the diagram, bounded by the curve and the x-axis.

Question 23

Core

Nov 2023 p31 q9 · Source ID 1784



The diagram shows the curve $y = xe^{-\frac{1}{4}x^2}$, for $x \geq 0$, and its maximum point M.

(a) Find the exact coordinates of M.

(b) Using the substitution $x = \sqrt{u}$, or otherwise, find by integration the exact area of the shaded region bounded by the curve, the x-axis and the line $x = 3$.

Question 24

Core

June 2023 p32 q9 · Source ID 2097

Let $f(x) = \frac{2x^2+17x-17}{(1+2x)(2-x)^2}$.

(a) Express $f(x)$ in partial fractions.

(b) Hence show that $\int_0^1 f(x)\,dx = \frac{5}{2} - \ln 72$.

TOPIC REVISION

Numerical Solutions of Equations

Paper 3 Skills

- Finding a starting point, improving a solution and iteration

Key Results

- A sign change, $f(a)f(b) < 0$, shows a root lies between a and b if f is continuous.
- For iteration $x_{n+1} = g(x_n)$, quote successive values to the required accuracy.
- Use sketches to justify number of roots before numerical refinement.

Common Traps

- Rounding intermediate iterations too early.
- Not showing enough iterations to justify the final accuracy.

Selected Practice

Question 25

Core

June 2023 p32 q6 · Source ID 1809

The equation $\cot \frac{1}{2}x = 3x$ has one root in the interval $0 < x < \pi$, denoted by α.

(a) Show by calculation that α lies between 0.5 and 1.

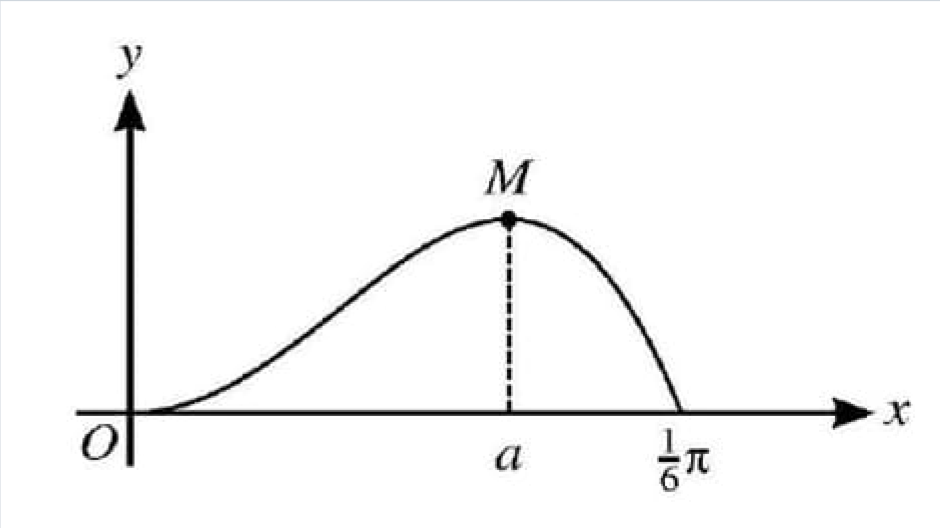
(b) Show that, if a sequence of positive values given by the iterative formula $x_{n+1} = \frac{1}{3}\left(x_n + 4\arctan\left(\frac{1}{3x_n}\right)\right)$ converges, then it converges to α.

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

Question 26

Exam

June 2023 p33 q5 · Source ID 1828



The diagram shows the part of the curve $y = x^2 \cos 3x$ for $0 \leq x \leq \frac{1}{6}\pi$, and its maximum point M, where $x = a$.

(a) Show that a satisfies the equation $a = \frac{1}{3}\arctan\left(\frac{2}{3a}\right)$.

(b) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

Question 27

Harder

June 2023 p31 q9 · Source ID 1847

The constant a is such that $\int_0^a xe^{-2x}\,dx = \frac{1}{8}$.

(a) Show that $a = \frac{1}{2}\ln(4a + 2)$.

(b) Verify by calculation that a lies between 0.5 and 1.

(c) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

TOPIC REVISION

Further Algebra

Paper 3 Skills

- Binomial expansion
- Partial fractions and binomial expansions

Key Results

- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$.
- State the validity condition, usually $|kx| < 1$.
- Use partial fractions first when expanding rational functions.

Common Traps

- Forgetting the range of validity.
- Expanding about the wrong small quantity, e.g. using $1 - 2x$ but treating it like $1 - x$.

Selected Practice

Question 28

Core

Nov 2022 p33 q2 · Source ID 2027

Expand $\sqrt{\frac{1+2x}{1-2x}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients.

Question 29

Exam

June 2022 p31 q2 · Source ID 2038

(a) Expand $(2 - x^2)^{-2}$ in ascending powers of x, up to and including the term in x^4, simplifying the coefficients.

(b) State the set of values of x for which the expansion is valid.

Question 30

Core

Nov 2023 p33 q9 · Source ID 2049

Let $f(x) = \frac{17x^2 - 7x + 16}{(2+3x^2)(2-x)}$.

(a) Express $f(x)$ in partial fractions.

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3.

(c) State the set of values of x for which the expansion in (b) is valid. Give your answer in an exact form.

Question 31

Exam

Nov 2023 p31 q10 · Source ID 2060

Let $f(x) = \frac{24x+13}{(1-2x)(2+x)^2}$.

(a) Express $f(x)$ in partial fractions.

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2.

(c) State the set of values of x for which the expansion in (b) is valid.

TOPIC REVISION

Complex Numbers

Paper 3 Skills

- Complex numbers and Argand diagrams

Key Results

- If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$ and $\arg z$ is measured from the positive real axis.
- $|z - a| = r$ is a circle centre a, radius r.
- Multiplying/dividing complex numbers is often easier after rationalising or using modulus/argument.

Common Traps

- Choosing the wrong quadrant for the argument.
- Forgetting that conjugation reflects in the real axis.

Selected Practice

Question 32

Core

June 2023 p32 q3 · Source ID 1906

Diagram / sketch space

Use the original exam diagram if available.

(a) On an Argand diagram, sketch the locus of points representing complex numbers z satisfying $|z + 3 - 2i| = 2$.

(b) Find the least value of $|z|$ for points on this locus, giving your answer in an exact form.

Question 33

Exam

June 2023 p31 q10 · Source ID 1917

The polynomial $x^3 + 5x^2 + 31x + 75$ is denoted by $p(x)$.

(a) Show that $(x + 3)$ is a factor of $p(x)$.

(b) Show that $z = -1 + 2\sqrt{6}i$ is a root of $p(z) = 0$.

(c) Hence find the complex numbers z which are roots of $p(z^2) = 0$.

Question 34

Harder

Mar 2023 p32 q4 · Source ID 1920

Solve the equation $\frac{5z}{1+2i} - zz^* + 30 + 10i = 0$, giving your answers in the form $x + iy$, where x and y are real.

Question 35

Harder

Feb/Mar 2023 p32 q2 · Source ID 1921

Diagram / sketch space

Use the original exam diagram if available.

(a) On an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities $-\frac{1}{3}\pi \leq \arg(z - 1 - 2i) \leq \frac{1}{3}\pi$ and $\operatorname{Re} z \leq 3$.

(b) Calculate the least value of $\arg z$ for points in the region from (a). Give your answer in radians correct to 3 decimal places.

TOPIC REVISION

Vectors

Paper 3 Skills

- Displacement, position vectors, scalar product and vector lines
- Intersection of two lines
- Vector review

Key Results

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$. Perpendicular vectors have dot product 0.
- A line can be written as $\mathbf{r} = \mathbf{a} + \lambda \mathbf{d}$.
- For line intersections, equate components and solve the parameters consistently.

Common Traps

- Confusing position vectors with displacement vectors.
- Assuming two 3D lines intersect just because two component equations work.

Selected Practice

Question 36

Core

Nov 2023 p33 q11 · Source ID 2126

The line l has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The points A and B have position vectors $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively.

(a) Find a unit vector in the direction of l.

The line m passes through the points A and B.

(b) Find a vector equation for m.

(c) Determine whether lines l and m are parallel, intersect or are skew.

Question 37

Exam

Nov 2023 p32 q10 · Source ID 2137

The equations of the lines l and m are given by

$$l: \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \text{ and } m: \mathbf{r} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix},$$

where c is a positive constant. It is given that the angle between l and m is $60°$.

(a) Find the value of c.

(b) Show that the length of the perpendicular from (6, -3, 6) to l is $\sqrt{11}$.

Question 38

Core

June 2023 p31 q6 · Source ID 2178

Relative to the origin O, the points A, B, and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

The quadrilateral $ABCD$ is a parallelogram.

(a) Find the position vector of D.

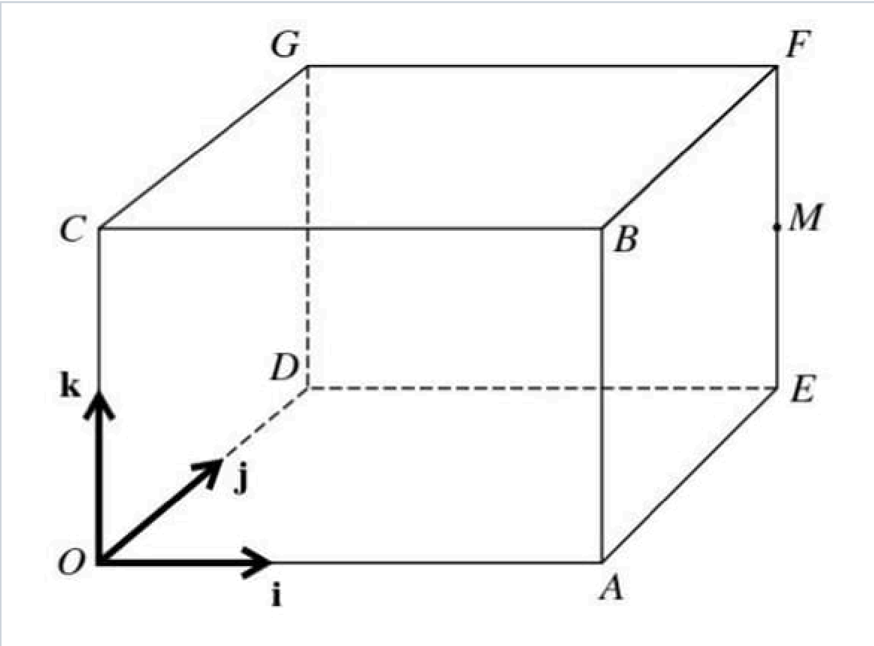
(b) The angle between BA and BC is θ. Find the exact value of $\cos\theta$.

(c) Hence find the area of $ABCD$, giving your answer in the form $p\sqrt{q}$, where p and q are integers.

Question 39

Core

Nov 2023 p31 q11 · Source ID 2231



In the diagram, *OABCDEFG* is a cuboid in which *OA = 3* units, *OC = 2* units and *OD = 2* units. Unit vectors *i, j* and *k* are parallel to *OA, OD* and *OC* respectively. *M* is the midpoint of *EF*.

(a) Find the position vector of *M*.

The position vector of *P* is *i + j + 2k*.

(b) Calculate angle *PAM*.

(c) Find the exact length of the perpendicular from *P* to the line passing through *O* and *M*.

TOPIC REVISION

Differential Equations

Paper 3 Skills

- Separating the variables
- Forming a differential equation from a problem
- Differential equations review

Key Results

- Separate variables before integrating: collect all y-terms with dy and x-terms with dx.
- Use the initial condition to find the constant of integration.
- Model questions often require forming $\frac{dV}{dt}$, $\frac{dy}{dx}$, or a proportionality equation first.

Common Traps

- Leaving the arbitrary constant after an initial condition is given.
- Integrating before the variables are fully separated.

Selected Practice

Question 40

Core

June 2023 p33 q8 · Source ID 2265

The variables x and y satisfy the differential equation $\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$ for $x > 0$. It is given that $x = 4$ when $y = 2\sqrt{3}$. Solve the differential equation to obtain the value of x when $y = 2$.

Question 41

Core

Nov 2022 p33 q10 · Source ID 2279

A gardener is filling an ornamental pool with water using a hose that delivers 30 litres of water per minute. Initially, the pool is empty. At time t minutes after filling begins, the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form

$$\frac{dV}{dt} = a - bV$$

(a) Write down the values of the constants a and b.

(b) Solve the differential equation and find the value of t when $V = 1000$.

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large.

Question 42

Core

Nov 2023 p32 q11 · Source ID 2324

The variables *x* and *y* satisfy the differential equation

$x^2 \frac{dy}{dx} + y^2 + y = 0$.

It is given that $x = 1$ when $y = 1$.

(a) Solve the differential equation to obtain an expression for *y* in terms of *x*.

(b) State what happens to the value of *y* when *x* tends to infinity. Give your answer in an exact form.

Question 43

Exam

Nov 2023 p33 q8 · Source ID 2340

The variables *x* and *y* satisfy the differential equation

$e^{4x} \frac{dy}{dx} = \cos^2 3y$.

It is given that $y = 0$ when $x = 2$.

Solve the differential equation, obtaining an expression for *y* in terms of *x*.

MIXED REVISION

Mixed Paper 3 Practice

These questions are from March 2026 p32 and are intended as a final mixed set.

Question 44

Mixed

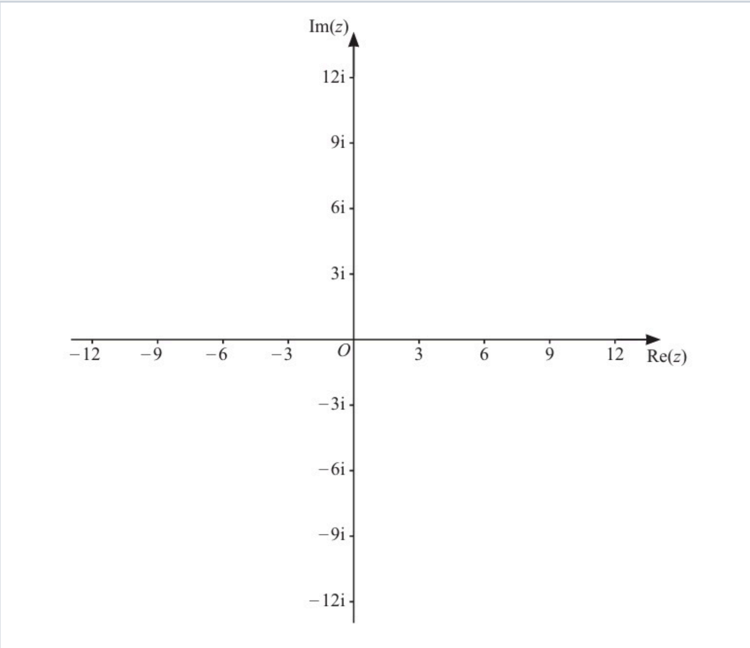
March 2026 p32 q1 · Source ID 4324

Solve the equation $4 \times 2^{x+2} - 5 \times 2^{2-x} = 3$. Give your answer correct to 3 significant figures.

Question 45

Mixed

March 2026 p32 q2 · Source ID 4325



The complex number z satisfies $|z| = 9$ and $\frac{1}{2}\pi \leq \arg z < \pi$.

(a) On the Argand diagram, sketch the locus of the points representing z.

(b) On the **same diagram**, sketch the locus of the points representing $z^* + 3$.

Question 46

Mixed

March 2026 p32 q3 · Source ID 4326

Find the exact value of

$$\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} 3 \sin x \sin 2x \, dx.$$

Give your answer in the form $p\sqrt{3} + q\sqrt{2}$, where p and q are rational.

Question 47

Mixed

March 2026 p32 q4 · Source ID 4327

The coefficient of x^3 in the expansion of $(1 - ax)^{\frac{2}{5}}$ is 1.

(a) Find the value of a.

(b) Hence, find the coefficient of x^4 in the expansion of $(2x + 1)(1 - ax)^{\frac{2}{5}}$.

(c) State the set of values of x for which the expansion in **4(b)** is valid.

Question 48

Mixed

March 2026 p32 q5 · Source ID 4328

It is given that $z = \frac{3+\lambda i}{\lambda+2i}$, where λ is a real constant.

(a) Find the value of λ for which $\arg z = \frac{1}{4}\pi$.

(b) When λ has the value found in **5(a)**, find the exact value of $|z|$, making your method clear.

Question 49

Mixed

March 2026 p32 q6 · Source ID 4329

The polynomial $2x^4 + ax^3 + 4x^2 + bx - 3$ is denoted by $p(x)$.

It is given that $(x^2 + x + 1)$ is a factor of $p(x)$.

(a) Find the values of a and b.

(b) Hence, show that $(x + 3)$ is a factor of $p(x)$.

Question 50

Mixed

March 2026 p32 q7 · Source ID 4330

(a) By sketching a suitable pair of graphs, show that the equation $\ln x = \operatorname{cosec} \frac{1}{2}x$ has exactly one root in the interval $0 < x < \pi$.

(b) Verify by calculation that this root lies between 2.6 and 2.9.

(c) Use the iterative formula $x_{n+1} = \exp\left(\operatorname{cosec} \frac{1}{2}x_n\right)$ to determine the root correct to 3 decimal places.

Give the result of each iteration to 5 decimal places.

[$\exp(x)$ is an alternative notation for e^x.]

Question 51

Mixed

March 2026 p32 q8 · Source ID 4331

The variables x and y satisfy the differential equation

$$ye^{3x}\frac{dy}{dx} = x(y + 5).$$

It is given that $y = 0$ when $x = 0$.

Solve the differential equation to obtain an equation in x and y.

Question 52

Mixed

March 2026 p32 q9 · Source ID 4332

Let

$$I = \int_1^3 \frac{x^3}{3 + x^2}\,dx.$$

(a) Using the substitution $x = \sqrt{3}\tan u$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 3\tan^3 u\,du$.

(b) Hence, or otherwise, find the exact value of I. Give your answer in the form $p + q\ln r$, where p, q and r are rational.

Question 53

Mixed

March 2026 p32 q10 · Source ID 4333

The variables x and y satisfy the equation $y^2 = k\frac{x-2}{x+2}$, where k is a constant.

(a) Show that $\frac{dy}{dx} = \frac{2y}{x^2-4}$.

(b) Given that $k = 5$, find the angle between the tangents to the curve when $x = 3$.

Give your answer in the form $a\tan^{-1}\left(\frac{b}{c}\right)$, where a, b and c are integers.

Question 54

Mixed

March 2026 p32 q11 · Source ID 4334

The points A and B have position vectors $\overrightarrow{OA} = 2i + 2j - k$ and $\overrightarrow{OB} = 4i + 2j + 4k$ relative to the origin, O.

(a) Show that the perpendicular distance from A to the line through O and B is $\frac{1}{3}\sqrt{65}$.

(b) The point C has position vector $\overrightarrow{OC} = 3i + pj + qk$, where p and q are constants.

Given that

- $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$ and
- angle $AOC =$ angle COB,

find the values of p and q.

ANSWER SECTION

Fully Worked Solutions

Solution to Question 1

Nov 2021 p32 q1

$$\frac{1}{\cos\theta} = 3\cos\theta + 1$$

$$1 = 3\cos^2\theta + \cos\theta$$

$$3\cos^2\theta + \cos\theta - 1 = 0$$

$$\cos\theta = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2 \times 3}$$

$$= \frac{-1 \pm \sqrt{13}}{6}$$

$\cos\theta \approx 0.4342$ or $\cos\theta \approx -0.7675$

For $\cos\theta = 0.4342$: $\theta = 64.3^\circ$ and $360^\circ - 64.3^\circ = 295.7^\circ$

For $\cos\theta = -0.7675$: $\theta = 140.1^\circ$ and $360^\circ - 140.1^\circ = 219.9^\circ$

$$\boxed{\theta = 64.3^\circ,\ 140.1^\circ,\ 219.9^\circ,\ 295.7^\circ}$$

Solution to Question 2

Feb/Mar 2022 p32 q5

Given:

$\tan(\alpha + \beta) = 2$

$\tan\alpha = 3\tan\beta$

Using the tangent addition formula:

$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 2$

Substitute $\tan\alpha = 3\tan\beta$:

$\frac{3\tan\beta + \tan\beta}{1 - 3\tan^2\beta} = 2$

Simplify:

$\frac{4\tan\beta}{1 - 3\tan^2\beta} = 2$

Cross-multiply:

$4\tan\beta = 2(1 - 3\tan^2\beta)$

$4\tan\beta = 2 - 6\tan^2\beta$

Rearrange to form a quadratic equation:

$6\tan^2\beta + 4\tan\beta - 2 = 0$

Divide by 2:

$3\tan^2\beta + 2\tan\beta - 1 = 0$

Let $x = \tan\beta$, solve:

$3x^2 + 2x - 1 = 0$

Using the quadratic formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 3 \cdot (-1)}}{2 \cdot 3}$

$x = \frac{-2 \pm \sqrt{4 + 12}}{6}$

$x = \frac{-2 \pm \sqrt{16}}{6}$

$x = \frac{-2 \pm 4}{6}$

$x = \frac{2}{6} = \frac{1}{3} \quad \text{or} \quad x = \frac{-6}{6} = -1$

Thus, $\tan\beta = \frac{1}{3}$ or $\tan\beta = -1$.

For $\tan\beta = \frac{1}{3}$, $\tan\alpha = 1$, so $\alpha = 45°$ and $\beta = 18.4°$.

For $\tan\beta = -1$, $\tan\alpha = -3$, so $\alpha = 108.4°$ and $\beta = 135°$.

Solution to Question 3

Nov 2021 p32 q8

(a) Expand $(\cos^2\theta + \sin^2\theta)^2$:

$(\cos^2\theta + \sin^2\theta)^2 = \cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta$

Since $\cos^2\theta + \sin^2\theta = 1$, we have:

$\cos^4\theta + \sin^4\theta = 1 - 2\cos^2\theta\sin^2\theta$

Using the double angle identity $\sin 2\theta = 2\sin\theta\cos\theta$, we get:

$\cos^4\theta + \sin^4\theta = 1 - \frac{1}{2}\sin^2 2\theta$

(b) Solve $\cos^4\theta + \sin^4\theta = \frac{5}{9}$:

Set $1 - \frac{1}{2}\sin^2 2\theta = \frac{5}{9}$

$\frac{1}{2}\sin^2 2\theta = 1 - \frac{5}{9} = \frac{4}{9}$

$\sin^2 2\theta = \frac{8}{9}$

$\sin 2\theta = \pm\frac{2\sqrt{2}}{3}$

Find 2θ:

$2\theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ or $2\theta = 180° - \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Calculate θ:

$\theta = 35.3°, 54.7°, 125.3°, 144.7°$

Solution to Question 4

Nov 2023 p31 q1

To prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$, we start by using trigonometric identities.

nn
First, express $\cos 4\theta$ and $\cos 2\theta$ using double angle formulas:

nn$\cos 4\theta = 2\cos^2 2\theta - 1nn\cos 2\theta = 2\cos^2\theta - 1$nn
Substitute $\cos 2\theta$ into the expression for $\cos 4\theta$:

nn$\cos 4\theta = 2(2\cos^2\theta - 1)^2 - 1$nn
Expand $(2\cos^2\theta - 1)^2$:

nn$(2\cos^2\theta - 1)^2 = 4\cos^4\theta - 4\cos^2\theta + 1$nn
Substitute back:

nn$\cos 4\theta = 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1 = 8\cos^4\theta - 8\cos^2\theta + 2 - 1 = 8\cos^4\theta - 8\cos^2\theta + 1$nn
Now substitute $\cos 4\theta$ and $\cos 2\theta$ into the left-hand side of the identity:

nn$\cos 4\theta - 4\cos 2\theta + 3 = (8\cos^4\theta - 8\cos^2\theta + 1) - 4(2\cos^2\theta - 1) + 3$nn
Simplify:

nn$= 8\cos^4\theta - 8\cos^2\theta + 1 - 8\cos^2\theta + 4 + 3nn= 8\cos^4\theta - 16\cos^2\theta + 8$nn
Now express in terms of $\sin^2\theta$:

nn$\cos^2\theta = 1 - \sin^2\theta$nn
Substitute:

nn$8\cos^4\theta - 16\cos^2\theta + 8 = 8(1 - \sin^2\theta)^2 - 16(1 - \sin^2\theta) + 8$nn
Expand:

nn$= 8(1 - 2\sin^2\theta + \sin^4\theta) - 16 + 16\sin^2\theta + 8nn= 8 - 16\sin^2\theta + 8\sin^4\theta - 16 + 16\sin^2\theta + 8nn= 8\sin^4\theta$nn
Thus, the identity is proven.

Solution to Question 5

June 2023 P33 q6

(a) To express $3\cos x + 2\cos(x - 60°)$ in the form $R\cos(x - \alpha)$:

First, expand $2\cos(x - 60°)$:

$2\cos(x - 60°) = 2(\cos x \cos 60° + \sin x \sin 60°) = \cos x + \sqrt{3}\sin x$
Combine with $3\cos x$:

$3\cos x + \cos x + \sqrt{3}\sin x = 4\cos x + \sqrt{3}\sin x$
Now, use the identity:

$a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$
where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$.

Here, $a = 4$ and $b = \sqrt{3}$:

$R = \sqrt{4^2 + (\sqrt{3})^2} = \sqrt{16 + 3} = \sqrt{19}$ $\tan\alpha = \frac{\sqrt{3}}{4}$
Thus, $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{4}\right) \approx 23.41°$.

(b) Solve $3\cos 2\theta + 2\cos(2\theta - 60°) = 2.5$:

Using the result from part (a), express:

$3\cos 2\theta + 2\cos(2\theta - 60°) = \sqrt{19}\cos(2\theta - 23.41°)$
Set equal to 2.5:

$\sqrt{19}\cos(2\theta - 23.41°) = 2.5$
$\cos(2\theta - 23.41°) = \frac{2.5}{\sqrt{19}}$

Find 2θ:

$2\theta - 23.41° = \cos^{-1}\left(\frac{2.5}{\sqrt{19}}\right)$
Calculate 2θ:

$2\theta = \cos^{-1}\left(\frac{2.5}{\sqrt{19}}\right) + 23.41°$
$\theta = 39.2°$ and $\theta = 82.1°$ (considering the range $0° < \theta < 180°$).

Solution to Question 6

June 2022 p33 q1

First, remove the modulus by considering the cases for the inequality:

1. Solve the inequality without modulus: $2(3x + a)^2 < (2x + 3a)^2$.

2. Expand both sides: $2(9x^2 + 6ax + a^2) < 4x^2 + 12ax + 9a^2$.

3. Simplify: $18x^2 + 12ax + 2a^2 < 4x^2 + 12ax + 9a^2$.

4. Rearrange: $14x^2 - 7a^2 < 0$.

5. Factorize: $14(x^2 - \frac{a^2}{2}) < 0$.

6. Solve for critical points: $x = \pm\frac{a}{\sqrt{2}}$.

7. Simplify critical points: $x = \frac{a}{2}$ and $x = -\frac{5}{8}a$.

8. Determine the interval: $-\frac{5}{8}a < x < \frac{1}{4}a$.

Solution to Question 7

Nov 2023 p32 q1

(a) The graph of $y = |4x - 2|$ is a V-shaped graph. The vertex of the graph is at $x = \frac{1}{2}$, $y = 0$. The graph is symmetrical about the vertical line $x = \frac{1}{2}$ and extends into the second quadrant.

(b) To solve $1 + 3x < |4x - 2|$, consider two cases:

Case 1: $4x - 2 \geq 0$ (i.e., $x \geq \frac{1}{2}$)

Then $|4x - 2| = 4x - 2$.

Solving $1 + 3x < 4x - 2$ gives:

$1 + 3x < 4x - 2$

$1 + 3x - 4x < -2$

$1 - x < -2$

$-x < -3$

$x > 3$

Case 2: $4x - 2 < 0$ (i.e., $x < \frac{1}{2}$)

Then $|4x - 2| = -(4x - 2) = -4x + 2$.

Solving $1 + 3x < -4x + 2$ gives:

$1 + 3x < -4x + 2$

$1 + 3x + 4x < 2$

$1 + 7x < 2$

$7x < 1$

$x < \frac{1}{2}$

Thus, the solution is $x < \frac{1}{2}$ or $x > 3$.

Solution to Question 8

Nov 2023 p33 q3

To find the values of a and b, we use the Remainder Theorem. For $p(x)$ divided by $(x+2)$, substitute $x=-2$ into $p(x)$:

$2(-2)^3+a(-2)^2+b(-2)+6=-38$

$-16+4a-2b+6=-38$

$4a-2b-10=-38$

$4a-2b=-28$

$2a-b=-14 \text{ (Equation 1)}$

For $p(x)$ divided by $(2x-1)$, substitute $x=\frac{1}{2}$ into $p(x)$:

$2\left(\frac{1}{2}\right)^3+a\left(\frac{1}{2}\right)^2+b\left(\frac{1}{2}\right)+6=\frac{19}{2}$

$\frac{1}{4}+\frac{a}{4}+\frac{b}{2}+6=\frac{19}{2}$

Multiply through by 4 to clear fractions:

$1+a+2b+24=38$

$a+2b=13 \text{ (Equation 2)}$

Now solve the system of equations:

Equation 1: $2a-b=-14$

Equation 2: $a+2b=13$

Multiply Equation 1 by 2:

$4a-2b=-28$

Add to Equation 2:

$4a-2b+a+2b=-28+13$

$5a=-15$

$a=-3$

Substitute $a=-3$ into Equation 2:

$-3+2b=13$

$2b=16$

$b=8$

Thus, $a=-3$ and $b=8$.

Solution to Question 9

Nov 2023 p32 q3

Since $p(x)$ is divisible by $(2x-1)$, substituting $x=\frac{1}{2}$ into $p(x)$ gives:

$\frac{1}{4}+\frac{a}{4}-\frac{11}{2}+b=0$

Multiplying through by 4 to clear fractions:

$1+a-22+4b=0$

$a+4b=21$

When $p(x)$ is divided by $(x+1)$, the remainder is 12. Substituting $x=-1$ into $p(x)$ gives:

$-2+a+11+b=12$

$a+b=3$

Solving the system of equations:

1. $a+4b=21$
2. $a+b=3$

Subtract equation 2 from equation 1:

$(a+4b)-(a+b)=21-3$

$3b=18$

$b=6$

Substitute $b=6$ into $a+b=3$:

$a+6=3$

$a=-3$

Thus, $a=-3$ and $b=6$.

Solution to Question 10

June 2022 p33 q3

(a) Start with the equation $\log_3(2x+1) = 1 + 2\log_3(x-1)$.

Using the law of logarithms, $2\log_3(x-1) = \log_3((x-1)^2)$.

Thus, the equation becomes $\log_3(2x+1) = \log_3(3) + \log_3((x-1)^2)$.

This simplifies to $\log_3(2x+1) = \log_3(3(x-1)^2)$.

Equating the arguments gives $2x+1 = 3(x-1)^2$.

Expanding $3(x-1)^2$ gives $3(x^2-2x+1) = 3x^2-6x+3$.

Thus, $2x+1 = 3x^2-6x+3$.

Rearranging gives $3x^2-8x+2 = 0$.

(b) Use the quadratic equation from part (a) to solve for y:

$3(2y)^2 - 8(2y) + 2 = 0$ simplifies to $12y^2 - 16y + 2 = 0$.

Divide through by 2: $6y^2 - 8y + 1 = 0$.

Using the quadratic formula $y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, where $a = 6, b = -8, c = 1$.

$y = \frac{8\pm\sqrt{(-8)^2-4\cdot6\cdot1}}{12}$.

$y = \frac{8\pm\sqrt{64-24}}{12}$.

$y = \frac{8\pm\sqrt{40}}{12}$.

$y = \frac{8\pm2\sqrt{10}}{12}$.

$y = \frac{4\pm\sqrt{10}}{6}$.

Calculating gives $y \approx 1.19$ (to 2 decimal places).

Solution to Question 11

Nov 2021 p33 q3

Start with the equation $4^{x-2} = 4^x - 4^2$.

Rewrite 4^{x-2} as $\frac{4^x}{4^2}$, so the equation becomes:

$\frac{4^x}{16} = 4^x - 16$.

Multiply through by 16 to clear the fraction:

$4^x = 16(4^x - 16)$.

Expand the right side:

$4^x = 16 \cdot 4^x - 256$.

Rearrange to isolate terms involving 4^x:

$4^x - 16 \cdot 4^x = -256$.

Factor out 4^x:

$4^x(1 - 16) = -256$.

Simplify:

$-15 \cdot 4^x = -256$.

Divide both sides by -15:

$4^x = \frac{256}{15}$.

Take the logarithm of both sides to solve for x:

$x \log(4) = \log\left(\frac{256}{15}\right)$.

Solve for x:

$x = \frac{\log\left(\frac{256}{15}\right)}{\log(4)}$.

Calculate the value:

$x \approx 2.047$.

Solution to Question 12

Nov 2023 p33 q1

First, consider the inequality $|2^{x+1} - 2| < 0.5$. This can be rewritten as two separate inequalities:

1. $2^{x+1} - 2 < 0.5$
2. $2^{x+1} - 2 > -0.5$

For the first inequality:

$2^{x+1} < 2.5$

Taking the logarithm base 2, we have:

$(x+1)\ln 2 < \ln 2.5$

$x < \frac{\ln 2.5}{\ln 2} - 1$

For the second inequality:

$2^{x+1} > 1.5$

Taking the logarithm base 2, we have:

$(x+1)\ln 2 > \ln 1.5$

$x > \frac{\ln 1.5}{\ln 2} - 1$

Calculating the critical values:

$x < \frac{\ln 2.5}{\ln 2} - 1 \approx 0.322$

$x > \frac{\ln 1.5}{\ln 2} - 1 \approx -0.415$

Thus, the solution is $-0.415 < x < 0.322$.

Solution to Question 13

Nov 2022 p33 q1

Start with the equation:

$\ln(2x-1) = 2\ln(x+1) - \ln x$

Apply the logarithm power rule: $2\ln(x+1) = \ln((x+1)^2)$

So the equation becomes:

$\ln(2x-1) = \ln((x+1)^2) - \ln x$

Use the logarithm quotient rule: $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

$\ln(2x-1) = \ln\left(\frac{(x+1)^2}{x}\right)$

Since the logarithms are equal, the arguments must be equal:

$2x - 1 = \frac{(x+1)^2}{x}$

Multiply both sides by x to clear the fraction:

$x(2x-1) = (x+1)^2$

Expand both sides:

$2x^2 - x = x^2 + 2x + 1$

Rearrange to form a quadratic equation:

$2x^2 - x - x^2 - 2x - 1 = 0$

$x^2 - 3x - 1 = 0$

Solve the quadratic equation using the quadratic formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1$, $b = -3$, $c = -1$

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$

$x = \frac{3 \pm \sqrt{9+4}}{2}$

$x = \frac{3 \pm \sqrt{13}}{2}$

Calculate the positive root:

$x = \frac{3 + \sqrt{13}}{2} \approx 3.303$

Solution to Question 14

Nov 2023 p31 q3

Given $y = ab^x$, taking natural logarithms gives $\ln y = \ln a + x \ln b$.

For the point (1, 3.7):

$3.7 = \ln a + 1 \cdot \ln b$

For the point (2.2, 6.46):

$6.46 = \ln a + 2.2 \cdot \ln b$

We have two equations:

1. $3.7 = \ln a + \ln b$
2. $6.46 = \ln a + 2.2 \ln b$

Subtract equation 1 from equation 2:

$6.46 - 3.7 = (\ln a + 2.2 \ln b) - (\ln a + \ln b)$

$2.76 = 1.2 \ln b$

$\ln b = \frac{2.76}{1.2} = 2.3$

$b = e^{2.3} \approx 9.97$

Substitute $\ln b = 2.3$ into equation 1:

$3.7 = \ln a + 2.3$

$\ln a = 3.7 - 2.3 = 1.4$

$a = e^{1.4} \approx 4.06$

Solution to Question 15

Nov 2023 p33 q7

(a) Differentiate the equation $x^3 + y^2 + 3x^2 + 3y = 4$ implicitly with respect to x:

$3x^2 + 2y\frac{dy}{dx} + 6x + 3\frac{dy}{dx} = 0$.

Rearrange to solve for $\frac{dy}{dx}$:

$2y\frac{dy}{dx} + 3\frac{dy}{dx} = -3x^2 - 6x$.

$\frac{dy}{dx}(2y+3) = -3x^2 - 6x$.

$\frac{dy}{dx} = -\frac{3x^2+6x}{2y+3}$.

(b) For the tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$.

Set the numerator of $\frac{dy}{dx} = -\frac{3x^2+6x}{2y+3}$ to zero:

$3x^2 + 6x = 0$.

Factorize:

$3x(x+2) = 0$.

$x = 0$ or $x = -2$.

Substitute $x = 0$ into the original equation:

$y^2 + 3y - 4 = 0$.

Factorize:

$(y-1)(y+4) = 0$.

$y = 1$ or $y = -4$.

Coordinates: $(0, 1)$ and $(0, -4)$.

Substitute $x = -2$ into the original equation:

$y^2 + 3y = 0$.

Factorize:

$y(y+3) = 0$.

$y = 0$ or $y = -3$.

Coordinates: $(-2, 0)$ and $(-2, -3)$.

Solution to Question 16

June 2023 p32 q7

(a) Differentiate the equation $3x^2 + 4xy + 3y^2 = 5$ implicitly with respect to x:

$\frac{d}{dx}(3x^2) = 6x$,

$\frac{d}{dx}(4xy) = 4x\frac{dy}{dx} + 4y$,

$\frac{d}{dx}(3y^2) = 6y\frac{dy}{dx}$.

Set the derivative equal to zero:

$6x + 4x\frac{dy}{dx} + 4y + 6y\frac{dy}{dx} = 0$.

Rearrange to solve for $\frac{dy}{dx}$:

$4x\frac{dy}{dx} + 6y\frac{dy}{dx} = -6x - 4y$,

$\frac{dy}{dx}(4x + 6y) = -6x - 4y$,

$\frac{dy}{dx} = -\frac{6x+4y}{4x+6y}$,

$\frac{dy}{dx} = -\frac{3x+2y}{2x+3y}$.

(b) The slope of the tangent is -2 (parallel to $y + 2x = 0$).

Set $\frac{dy}{dx} = -2$:

$-\frac{3x+2y}{2x+3y} = -2$,

$3x + 2y = 4x + 6y$,

$x = -4y$ or $y = -\frac{x}{4}$.

Substitute $x = -4y$ into the curve equation:

$3(-4y)^2 + 4(-4y)y + 3y^2 = 5$,

$48y^2 - 16y^2 + 3y^2 = 5$,

$35y^2 = 5$,

$y^2 = \frac{1}{7}$,

$y = \pm\frac{1}{\sqrt{7}}$.

For $y = \frac{1}{\sqrt{7}}$, $x = -4\left(\frac{1}{\sqrt{7}}\right) = -\frac{4}{\sqrt{7}}$.

For $y = -\frac{1}{\sqrt{7}}$, $x = 4\left(\frac{1}{\sqrt{7}}\right) = \frac{4}{\sqrt{7}}$.

The coordinates are $\left(\frac{4}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ and $\left(-\frac{4}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right)$.

Solution to Question 17

Nov 2023 p32 q2

To find the gradient $\frac{dy}{dx}$, we first find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$x = (\ln t)^2$ implies $\frac{dx}{dt} = 2\ln t \cdot \frac{1}{t} = \frac{2\ln t}{t}$.

$y = e^{2-t^2}$ implies $\frac{dy}{dt} = -2te^{2-t^2}$.

The gradient $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2te^{2-t^2}}{\frac{2\ln t}{t}} = -t^2e^{2-t^2} \cdot \frac{t}{2\ln t} = -\frac{t^3e^{2-t^2}}{2\ln t}$.

Substitute $t = e$:

$\frac{dy}{dx} = -\frac{e^3e^{2-e^2}}{2\ln e} = -\frac{e^3e^{2-e^2}}{2}$.

Simplifying gives $-e^{4-e^2}$.

Solution to Question 18

Nov 2023 p31 q1

To find the points where the gradient of the tangent is 8, we first need to find the derivative of $y = \frac{x^2}{1-3x}$.

Using the quotient rule, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, where $u = x^2$ and $v = 1 - 3x$.

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -3$.

Thus, the derivative $\frac{dy}{dx} = \frac{(1-3x)(2x)-x^2(-3)}{(1-3x)^2} = \frac{2x-6x^2+3x^2}{(1-3x)^2} = \frac{2x-3x^2}{(1-3x)^2}$.

Set $\frac{2x-3x^2}{(1-3x)^2} = 8$ and solve for x.

$2x - 3x^2 = 8(1 - 3x)^2$

$2x - 3x^2 = 8(1 - 6x + 9x^2)$

$2x - 3x^2 = 8 - 48x + 72x^2$

$75x^2 - 50x + 8 = 0$

Factor the quadratic: $(15x - 4)(5x - 2) = 0$

Solutions for x are $x = \frac{2}{5}$ and $x = \frac{4}{15}$.

Substitute back to find y:

For $x = \frac{2}{5}$, $y = \frac{(\frac{2}{5})^2}{1-3(\frac{2}{5})} = \frac{\frac{4}{25}}{1-\frac{6}{5}} = \frac{\frac{4}{25}}{-\frac{1}{5}} = -\frac{4}{5}$.

For $x = \frac{4}{15}$, $y = \frac{(\frac{4}{15})^2}{1-3(\frac{4}{15})} = \frac{\frac{16}{225}}{1-\frac{12}{15}} = \frac{\frac{16}{225}}{\frac{1}{5}} = \frac{16}{45}$.

Thus, the points are $\left(\frac{2}{5}, -\frac{4}{5}\right)$ and $\left(\frac{4}{15}, \frac{16}{45}\right)$.

Solution to Question 19

Nov 2022 p32 q3

To find the stationary point, we need to differentiate $y = \sin x \sin 2x$ and set the derivative to zero.

Using the product rule, $y' = \cos x \sin 2x + 2\sin x \cos 2x$.

Using the double angle formula $\sin 2x = 2\sin x \cos x$, we can express the derivative in terms of $\sin x$ and $\cos x$.

Equating the derivative to zero, we obtain $3\sin^2 x = 2$, which simplifies to $\tan^2 x = 2$.

Solving for x, we find $x = 0.955$ (to 3 significant figures).

Solution to Question 20

Nov 2021 p32 q6

(a) The expansion of $\sin(3x+2x)$ is $\sin 5x = \sin 3x \cos 2x + \cos 3x \sin 2x$.

The expansion of $\sin(3x-2x)$ is $\sin x = \sin 3x \cos 2x - \cos 3x \sin 2x$.

Adding these, we get $\sin 5x + \sin x = 2\sin 3x \cos 2x$.

Thus, $\frac{1}{2}(\sin 5x + \sin x) = \sin 3x \cos 2x$.

(b) We have $\int \sin 3x \cos 2x \, dx = \frac{1}{2}\int(\sin 5x + \sin x)\, dx$.

This integrates to $\frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right)$.

Evaluating from 0 to $\frac{1}{4}\pi$, we get:

$\left[-\frac{1}{10}\cos 5x - \frac{1}{2}\cos x\right]_0^{\frac{1}{4}\pi}$.

Substituting the limits, we find:

$-\frac{1}{10}\cos\left(\frac{5}{4}\pi\right) - \frac{1}{2}\cos\left(\frac{1}{4}\pi\right) + \frac{1}{10}\cos 0 + \frac{1}{2}\cos 0$.

This simplifies to $\frac{1}{5}(3-\sqrt{2})$.

Solution to Question 21

June 2022 p31 q6

(a) Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\, d\theta$.

Substitute into the integral:

$I = \int_0^3 \frac{27}{(9+x^2)^2}\, dx = \int_0^{\frac{\pi}{4}} \frac{27}{(9+9\tan^2\theta)^2} \cdot 3\sec^2\theta\, d\theta$.

$= \int_0^{\frac{\pi}{4}} \frac{81\sec^2\theta}{81\sec^4\theta}\, d\theta$.

$= \int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta$.

(b) The integral $\int \cos^2\theta\, d\theta$ can be expressed as $\int \frac{1+\cos 2\theta}{2}\, d\theta$.

$= \frac{1}{2}\int(1+\cos 2\theta)\, d\theta$.

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$.

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - (0+0)\right]$.

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right]$.

$= \frac{1}{8}(\pi+2)$.

Solution to Question 22

Nov 2023 p33 q10

(a) To find the equation of the tangent, we first need the derivative of $y = x\cos 2x$. Using the product rule:

$\frac{dy}{dx} = \frac{d}{dx}(x) \cdot \cos 2x + x \cdot \frac{d}{dx}(\cos 2x)$

$= \cos 2x - 2x\sin 2x$

At $x = \frac{1}{2}\pi$, $y = x\cos 2x = \frac{1}{2}\pi \cdot \cos\pi = -\frac{\pi}{2}$.

$\frac{dy}{dx} = \cos\pi - 2 \cdot \frac{1}{2}\pi \cdot \sin\pi = -1$.

The equation of the tangent is $y - (-\frac{\pi}{2}) = -1(x - \frac{1}{2}\pi)$.

Simplifying gives $x + y = 0$.

(b) To find the area of the shaded region, integrate $y = x\cos 2x$ from $x = 0$ to $x = \frac{\pi}{4}$.

Using integration by parts, let $u = x$ and $dv = \cos 2x\,dx$, then $du = dx$ and $v = \frac{1}{2}\sin 2x$.

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x$.

Evaluate from $x = 0$ to $x = \frac{\pi}{4}$:

$\left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{2} \cdot \frac{\pi}{4} \cdot \sin\frac{\pi}{2} + \frac{1}{4} \cdot \cos\frac{\pi}{2}\right) - \left(0 + \frac{1}{4} \cdot 1\right)$

$= \frac{\pi}{8} - \frac{1}{4}$.

Solution to Question 23

Nov 2023 p31 q9

(a) To find the coordinates of M, we need to find the maximum point of the function $y = xe^{-\frac{1}{4}x^2}$. Differentiate y with respect to x:

$\frac{dy}{dx} = e^{-\frac{1}{4}x^2} - \frac{x^2}{2}e^{-\frac{1}{4}x^2}$.

Set $\frac{dy}{dx} = 0$ to find critical points:

$e^{-\frac{1}{4}x^2}(1 - \frac{x^2}{2}) = 0$.

Since $e^{-\frac{1}{4}x^2} \neq 0$, solve $1 - \frac{x^2}{2} = 0$:

$x^2 = 2 \Rightarrow x = \sqrt{2}$.

Substitute $x = \sqrt{2}$ back into the original equation to find y:

$y = \sqrt{2}e^{-\frac{1}{4}(\sqrt{2})^2} = \sqrt{2}e^{-\frac{1}{2}}$.

Thus, the coordinates of M are $\left(\sqrt{2}, \sqrt{2}e^{-\frac{1}{2}}\right)$.

(b) To find the area under the curve from $x = 0$ to $x = 3$, use the substitution $x = \sqrt{u}$, so $dx = \frac{1}{2\sqrt{u}}du$.

The integral becomes:

$\int xe^{-\frac{1}{4}x^2}\,dx = \int \frac{1}{2}e^{-\frac{1}{4}u}\,du$.

Change the limits: when $x = 0, u = 0$ and when $x = 3, u = 9$.

Integrate:

$\frac{1}{2}\int e^{-\frac{1}{4}u}\,du = -2e^{-\frac{1}{4}u}\Big|_0^9$.

Calculate the definite integral:

$-2e^{-\frac{9}{4}} + 2e^0 = 2 - 2e^{-\frac{9}{4}}$.

Thus, the exact area of the shaded region is $2 - 2e^{-\frac{9}{4}}$.

Solution to Question 24

June 2023 p32 q9

(a) Express $f(x)$ in the form:

$\frac{A}{1+2x} + \frac{B}{2-x} + \frac{C}{(2-x)^2}$

Using partial fraction decomposition, equate:

$2x^2 + 17x - 17 = A(2-x)^2 + B(1+2x)(2-x) + C(1+2x)$

Expand and compare coefficients to find:

$A = -4, B = -3, C = 5.$

(b) Integrate each term separately:

$\int \frac{-4}{1+2x}\,dx = -2\ln(1+2x)$

$\int \frac{-3}{2-x}\,dx = 3\ln(2-x)$

$\int \frac{5}{(2-x)^2}\,dx = \frac{5}{2-x}$

Substitute limits from 0 to 1:

$-2\ln(3) + 3\ln(1) + \frac{5}{1} - (-2\ln(1) + 3\ln(2) + 5/2)$

Simplify to obtain:

$\frac{5}{2} - \ln 72.$

Solution to Question 25

June 2023 p32 q6

(a) Calculate $\cot\frac{1}{2}x - 3x$ at $x = 0.5$ and $x = 1$.

For $x = 0.5$, $\cot\frac{1}{2} \times 0.5 = \cot 0.25 \approx 3.92$ and $3 \times 0.5 = 1.5$. So, $3.92 - 1.5 > 0$.

For $x = 1$, $\cot\frac{1}{2} \times 1 = \cot 0.5 \approx 1.83$ and $3 \times 1 = 3$. So, $1.83 - 3 < 0$.

Since there is a change of sign, α lies between 0.5 and 1.

(b) Rearrange $\cot\frac{x}{2} = 3x$ to $x = \frac{1}{3}\left(x + 4\arctan\left(\frac{1}{3x}\right)\right)$.

The iterative formula $x_{n+1} = \frac{1}{3}\left(x_n + 4\arctan\left(\frac{1}{3x_n}\right)\right)$ is derived from this rearrangement.

If the sequence converges, it must converge to the root α.

(c) Start with an initial guess, say $x_0 = 0.5$.

Calculate successive iterations:

$x_1 = \frac{1}{3}\left(0.5 + 4\arctan\left(\frac{1}{3\times 0.5}\right)\right) \approx 0.7623$

$x_2 = \frac{1}{3}\left(0.7623 + 4\arctan\left(\frac{1}{3\times 0.7623}\right)\right) \approx 0.8037$

$x_3 = \frac{1}{3}\left(0.8037 + 4\arctan\left(\frac{1}{3\times 0.8037}\right)\right) \approx 0.7921$

$x_4 = \frac{1}{3}\left(0.7921 + 4\arctan\left(\frac{1}{3\times 0.7921}\right)\right) \approx 0.7951$

$x_5 = \frac{1}{3}\left(0.7951 + 4\arctan\left(\frac{1}{3\times 0.7951}\right)\right) \approx 0.7943$

$x_6 = \frac{1}{3}\left(0.7943 + 4\arctan\left(\frac{1}{3\times 0.7943}\right)\right) \approx 0.7945$

The value converges to 0.79 to 2 decimal places.

Solution to Question 26

June 2023 p33 q5

(a) To find the maximum point, we need to differentiate $y = x^2 \cos 3x$ and set the derivative to zero. Using the product rule:

$\frac{d}{dx}(x^2 \cos 3x) = \frac{d}{dx}(x^2) \cdot \cos 3x + x^2 \cdot \frac{d}{dx}(\cos 3x)$.

$= 2x \cos 3x - 3x^2 \sin 3x$.

Setting the derivative to zero for maximum point:

$2x \cos 3x - 3x^2 \sin 3x = 0$.

$2x \cos 3x = 3x^2 \sin 3x$.

$\frac{2}{3x} = \tan 3x$.

$x = \frac{1}{3}\arctan\left(\frac{2}{3x}\right)$.

At maximum point $x = a$, so $a = \frac{1}{3}\arctan\left(\frac{2}{3a}\right)$.

(b) Using the iterative formula $a_{n+1} = \frac{1}{3}\arctan\left(\frac{2}{3a_n}\right)$, start with an initial guess and iterate:

1. $a_1 = 0.5$
2. $a_2 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.5}\right) \approx 0.3435$
3. $a_3 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.3435}\right) \approx 0.3826$
4. $a_4 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.3826}\right) \approx 0.4264$
5. $a_5 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.4264}\right) \approx 0.4740$
6. $a_6 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.4740}\right) \approx 0.3017$
7. $a_7 = \frac{1}{3}\arctan\left(\frac{2}{3\times0.3017}\right) \approx 0.3989$

Continue iterating until the value stabilizes to 2 decimal places: $a \approx 0.36$.

Solution to Question 27

June 2023 p31 q9

(a) To solve $\int_0^a xe^{-2x}\,dx = \frac{1}{8}$, we start by integrating by parts. Let $u = x$ and $dv = e^{-2x}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-2x}$.

Using integration by parts, $\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$.

Integrating $\int e^{-2x}\,dx$ gives $-\frac{1}{2}e^{-2x}$, so $\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$.

Evaluating from 0 to a, we have:

$\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^a = -\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} + \frac{1}{4}$.

Setting this equal to $\frac{1}{8}$, we solve:

$-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} + \frac{1}{4} = \frac{1}{8}$.

Rearranging gives $-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} = -\frac{1}{8}$.

Multiplying through by -1 and simplifying, we find $a = \frac{1}{2}\ln(4a+2)$.

(b) To verify a lies between 0.5 and 1, calculate:

For $a = 0.5$, $\frac{1}{2}\ln(4\times 0.5+2) = \frac{1}{2}\ln(4) = 0.693\ldots$.

For $a = 1$, $\frac{1}{2}\ln(4\times 1+2) = \frac{1}{2}\ln(6) = 0.895\ldots$

Since $0.5 < 0.693 < 1$, a is between 0.5 and 1.

(c) Using the iterative formula $a_{n+1} = \frac{1}{2}\ln(4a_n+2)$, start with $a_0 = 0.75$:

$a_1 = \frac{1}{2}\ln(4\times 0.75+2) = 0.8047$

$a_2 = \frac{1}{2}\ln(4\times 0.8047+2) = 0.8261$

$a_3 = \frac{1}{2}\ln(4\times 0.8261+2) = 0.8343$

$a_4 = \frac{1}{2}\ln(4\times 0.8343+2) = 0.8373$

$a_5 = \frac{1}{2}\ln(4\times 0.8373+2) = 0.8385$

Continuing this process, we find $a \approx 0.84$ to 2 decimal places.

Solution to Question 28

Nov 2022 p33 q2

To expand $\sqrt{\frac{1+2x}{1-2x}}$, we can write it as $(1+2x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}}$.

First, expand $(1+2x)^{\frac{1}{2}}$ using the binomial series:

$1 + \frac{1}{2}(2x) + \frac{1}{2}\cdot\frac{-1}{2}(2x)^2 = 1 + x - x^2$.

Next, expand $(1-2x)^{-\frac{1}{2}}$:

$1 + \frac{1}{2}(2x) + \frac{1}{2}\cdot\frac{3}{2}(2x)^2 = 1 + x + \frac{3}{2}x^2$.

Now, multiply the expansions:

$(1 + x - x^2)(1 + x + \frac{3}{2}x^2)$.

Calculate the product up to x^2:

$1 + x + \frac{3}{2}x^2 + x + x^2 - x^3 - x^2 - x^3$.

Combine like terms:

$1 + 2x + 2x^2$.

Solution to Question 29

June 2022 p31 q2

(a) We start by rewriting $(2-x^2)^{-2}$ as $\left(\frac{1}{2}(1-\frac{x^2}{2})\right)^{-2}$.

Using the binomial expansion for $(1+u)^n$, where $n=-2$ and $u=-\frac{x^2}{2}$, we have:

$1+nu+\frac{n(n-1)}{2!}u^2+\cdots$

Substitute $n=-2$ and $u=-\frac{x^2}{2}$:

$1-2\left(-\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2}\left(-\frac{x^2}{2}\right)^2+\cdots$

$=1+x^2+\frac{3}{4}x^4+\cdots$

Multiply by $\frac{1}{4}$ to account for the factor $\frac{1}{4}$ from $\left(\frac{1}{2}\right)^{-2}$:

$\frac{1}{4}(1+x^2+\frac{3}{4}x^4)=\frac{1}{4}+\frac{1}{4}x^2+\frac{3}{16}x^4$

(b) The expansion is valid for $|x|<\sqrt{2}$ because the binomial expansion is valid for $|u|<1$, where $u=\frac{x^2}{2}$, leading to $|x^2/2|<1$, or $|x|<\sqrt{2}$.

Solution to Question 30

Nov 2023 p33 q9

(a) Express $f(x)$ in partial fractions:

Assume $\frac{17x^2-7x+16}{(2+3x^2)(2-x)}=\frac{Ax+B}{2+3x^2}+\frac{C}{2-x}$.

Multiply through by the denominator: $17x^2-7x+16=(Ax+B)(2-x)+C(2+3x^2)$.

Expand and equate coefficients to solve for A, B, and C:

$A=-2, B=3, C=5$.

Thus, $f(x)=\frac{-2x+3}{2+3x^2}+\frac{5}{2-x}$.

(b) Expand $f(x)$ in ascending powers of x:

Use the binomial expansion for $(2-x)^{-1}$ and $(2+3x^2)^{-1}$:

$(2-x)^{-1}=\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}$.

$(2+3x^2)^{-1}=\frac{1}{2}-\frac{3x^2}{4}+\frac{9x^4}{8}$.

Substitute back into the partial fractions and expand:

$\frac{-2x+3}{2+3x^2}=(-2x+3)\left(\frac{1}{2}-\frac{3x^2}{4}\right)$.

$\frac{5}{2-x}=5\left(\frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}\right)$.

Combine and simplify to get:

$4+\frac{1}{4}x-\frac{13}{8}x^2+\frac{29}{16}x^3$.

(c) The expansion is valid for $|x|<\sqrt[3]{\frac{2}{3}}$.

Solution to Question 31

Nov 2023 p31 q10

(a) Express $f(x)$ in partial fractions:

Assume $\frac{24x+13}{(1-2x)(2+x)^2} = \frac{A}{1-2x} + \frac{B}{2+x} + \frac{C}{(2+x)^2}$.

Multiply through by the denominator $(1-2x)(2+x)^2$ to clear the fractions:

$24x + 13 = A(2+x)^2 + B(1-2x)(2+x) + C(1-2x)$.

Expand and equate coefficients to find $A = 4$, $B = 2$, $C = -7$.

(b) Expand each partial fraction:

$\frac{A}{1-2x} = 4(1-2x)^{-1} \approx 4(1 + 2x + 4x^2)$.
$\frac{B}{2+x} = 2(2+x)^{-1} \approx 2(1 - \frac{x}{2} + \frac{x^2}{4})$.
$\frac{C}{(2+x)^2} = -7(2+x)^{-2} \approx -7(\frac{1}{4} - \frac{x}{4} + \frac{3x^2}{16})$.

Combine these to get:

$f(x) \approx \frac{13}{4} + \frac{37}{4}x + \frac{239}{16}x^2$.

(c) The expansion is valid for $|x| < \frac{1}{2}$.

Solution to Question 32

June 2023 p32 q3

(a) The equation $|z + 3 - 2i| = 2$ represents a circle on the Argand diagram with center at $(-3, 2)$ and radius 2. To sketch this, plot the center at $(-3, 2)$ and draw a circle with radius 2.

(b) To find the least value of $|z|$, calculate the distance from the origin $(0, 0)$ to the center $(-3, 2)$, which is $\sqrt{(-3)^2 + 2^2} = \sqrt{13}$. The least value of $|z|$ is the distance from the origin to the nearest point on the circle, which is $\sqrt{13} - 2$.

Solution to Question 33

June 2023 p31 q10

(a) Substitute $x = -3$ into $p(x)$:

$p(-3) = (-3)^3 + 5(-3)^2 + 31(-3) + 75 = -27 + 45 - 93 + 75 = 0$.

Since $p(-3) = 0$, $(x + 3)$ is a factor of $p(x)$.

(b) Substitute $z = -1 + 2\sqrt{6}i$ into $p(z)$:

Calculate $z^2 = (-1 + 2\sqrt{6}i)^2 = -1 - 4\sqrt{6}i - 24 = -25 - 4\sqrt{6}i$.

Calculate $z^3 = z \cdot z^2 = (-1 + 2\sqrt{6}i)(-25 - 4\sqrt{6}i) = 25 + 4\sqrt{6}i + 50\sqrt{6}i + 48 = 73 + 54\sqrt{6}i$.

Substitute into $p(z)$:

$p(z) = z^3 + 5z^2 + 31z + 75 = 73 + 54\sqrt{6}i + 5(-25 - 4\sqrt{6}i) + 31(-1 + 2\sqrt{6}i) + 75 = 0$.

Thus, $z = -1 + 2\sqrt{6}i$ is a root of $p(z) = 0$.

(c) Let $z^2 = w$, then $p(w) = w^3 + 5w^2 + 31w + 75 = 0$.

The roots of $p(w) = 0$ are $w_1 = 3i, w_2 = -3i, w_3 = -1 + 2\sqrt{6}i$.

For $w_1 = 3i$, $z = \pm\sqrt{3}i$.

For $w_2 = -3i$, $z = \pm\sqrt{6}i$.

For $w_3 = -1 + 2\sqrt{6}i$, $z = -1 \pm 2\sqrt{6}i$.

Thus, the roots are $z = \pm\sqrt{3}i, \pm\sqrt{6}i, -1 \pm 2\sqrt{6}i$.

Solution to Question 34

Mar 2023 p32 q4

Let $z = x + iy$ and $z^* = x - iy$. Substitute into the equation:

$\frac{5(x+iy)}{1+2i} - (x + iy)(x - iy) + 30 + 10i = 0$

Multiply numerator and denominator by the conjugate of the denominator:

$\frac{5(x+iy)(1-2i)}{(1+2i)(1-2i)} - (x^2 + y^2) + 30 + 10i = 0$

$(1 + 2i)(1 - 2i) = 1 + 4 = 5$

$\frac{5(x+iy)(1-2i)}{5} - x^2 - y^2 + 30 + 10i = 0$

$5(x + iy)(1 - 2i) = 5x - 10y + i(5y + 10x)$

$x - 2ix + iy + 2y - x^2 - y^2 + 30 + 10i = 0$

Equate real and imaginary parts:

Real: $x + 2y - x^2 - y^2 + 30 = 0$

Imaginary: $-2x + y + 10 = 0$

Solve the quadratic equations:

$x^2 - 9x + 18 = 0$ gives $x = 3$ or $x = 6$

$y^2 + 2y - 8 = 0$ gives $y = -4$ or $y = 2$

Thus, the solutions are $3 - 4i$ and $6 + 2i$.

Solution to Question 35

Feb/Mar 2023 p32 q2

(a) The region is defined by the argument inequalities $-\frac{1}{3}\pi \leq \arg(z - 1 - 2i) \leq \frac{1}{3}\pi$, which represent half-lines from the point $(1, 2)$ on the Argand diagram. These lines are symmetrical about the line $y = 2$ and are drawn between angles $\frac{\pi}{4}$ and $\frac{5\pi}{12}$. Additionally, the line $x = 3$ is drawn, extending in both quadrants. The correct region is shaded between these lines.

(b) To find the least value of $\arg z$, we use the formula $-\arctan\left(\frac{2\sqrt{3}-2}{3}\right)$. Calculating this gives -0.454 radians, correct to 3 decimal places.

Solution to Question 36

Nov 2023 p33 q11

The line *l* has equation $\mathbf{r} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The points *A* and *B* have position vectors $-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively.

(a) The direction vector of line *l* is $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$. The magnitude of this vector is $\sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$. Therefore, a unit vector in the direction of *l* is $\dfrac{1}{\sqrt{6}}(-\mathbf{i} + \mathbf{j} + 2\mathbf{k})$.

(b) The vector from *A* to *B* is $(3\mathbf{i} - \mathbf{j} + \mathbf{k}) - (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. rnThus, a vector equation for line *m* is $\mathbf{r} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \mu(5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$.

(c) The direction vector of *l* is $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and for *m* is $5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. These vectors are not scalar multiples of each other, so the lines are not parallel. Solving the system obtained by equating components of the general points on *l* and *m* does not yield a consistent solution, indicating the lines do not intersect. Therefore, lines *l* and *m* are skew.

Solution to Question 37

Nov 2023 p32 q10

(a) To find the value of c, we use the fact that the angle between the lines is $60°$. The direction vectors of l and m are $\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}-2\\4\\c\end{pmatrix}$ respectively. The cosine of the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by:

$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$

Substituting the given vectors and $\theta = 60°$, we have:

$\cos 60° = \frac{1(-2)+1(4)+2c}{\sqrt{1^2+1^2+2^2}\sqrt{(-2)^2+4^2+c^2}}$

$\frac{2+2c}{\sqrt{6}\sqrt{20+c^2}} = \frac{1}{2}$

Solving for c, we get:

$2+2c = \frac{1}{2}\sqrt{120+6c^2}$

$4+4c = \sqrt{120+6c^2}$

Squaring both sides:

$16+16c+16c^2 = 120+6c^2$

$10c^2+16c-104 = 0$

Solving this quadratic equation gives $c = 2$.

(b) To find the length of the perpendicular from the point (6, -3, 6) to the line l, consider a general point on l given by $\begin{pmatrix}3+\lambda\\-2+\lambda\\1+2\lambda\end{pmatrix}$. The vector from (6, -3, 6) to this point is:

$\begin{pmatrix}3+\lambda-6\\-2+\lambda+3\\1+2\lambda-6\end{pmatrix} = \begin{pmatrix}-3+\lambda\\1+\lambda\\-5+2\lambda\end{pmatrix}$

For the perpendicular, the dot product with the direction vector of l must be zero:

$(-3+\lambda)\cdot 1 + (1+\lambda)\cdot 1 + (-5+2\lambda)\cdot 2 = 0$

$-3+\lambda+1+\lambda-10+4\lambda = 0$

$6\lambda - 12 = 0$

$\lambda = 2$

The perpendicular vector is $\begin{pmatrix}-1\\3\\-1\end{pmatrix}$. Its length is:

$\sqrt{(-1)^2+3^2+(-1)^2} = \sqrt{11}$

Solution to Question 38

June 2023 p31 q6

(a) To find $\overrightarrow{OD}$, use the property of a parallelogram: $\overrightarrow{AB} = \overrightarrow{CD}$. Calculate $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

Then, $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$.

(b) To find $\cos\theta$, use the dot product formula: $\overrightarrow{BA} \cdot \overrightarrow{BC} = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\| \cos\theta$.

Calculate $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$.

Calculate $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix}$.

Dot product: $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-2)(-1) + (-2)(-5) + (1)(-6) = 2 + 10 - 6 = 6$.

Magnitudes: $\|\overrightarrow{BA}\| = \sqrt{(-2)^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$, $\|\overrightarrow{BC}\| = \sqrt{(-1)^2 + (-5)^2 + (-6)^2} = \sqrt{62}$.

$\cos\theta = \frac{6}{3\times\sqrt{62}} = \frac{2}{\sqrt{62}}$.

(c) The area of parallelogram $ABCD$ is given by $\text{Area} = \|\overrightarrow{BA}\|\|\overrightarrow{BC}\| \sin\theta$.

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{2}{\sqrt{62}}\right)^2} = \sqrt{\frac{58}{62}}$.

$\text{Area} = 3 \times \sqrt{62} \times \sqrt{\frac{58}{62}} = 3\sqrt{58}$.

Solution to Question 39

Nov 2023 p31 q11

(a) The position vector of M is the midpoint of EF. Since E is at *(3, 2, 0)* and F is at *(3, 2, 2)*, the midpoint M is at *(3, 2, 1)*. Therefore, the position vector of M is $\mathbf{3i} + \mathbf{2j} + \mathbf{k}$.

(b) To calculate angle PAM, find vectors $\overrightarrow{AM}$ and $\overrightarrow{AP}$:

$\overrightarrow{AM} = (\mathbf{3i} + \mathbf{2j} + \mathbf{k}) - (\mathbf{3i}) = \mathbf{2j} + \mathbf{k}$

$\overrightarrow{AP} = (\mathbf{i} + \mathbf{j} + \mathbf{2k}) - (\mathbf{3i}) = -\mathbf{2i} + \mathbf{j} + \mathbf{2k}$

Calculate the dot product: $\overrightarrow{AM} \cdot \overrightarrow{AP} = 0 + 2 \times 1 + 1 \times 2 = 4$

Find the magnitudes: $|\overrightarrow{AM}| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $|\overrightarrow{AP}| = \sqrt{(-2)^2 + 1^2 + 2^2} = \sqrt{9} = 3$

Use the cosine formula: $\cos\theta = \frac{4}{\sqrt{5} \times 3}$

$\theta = \cos^{-1}\left(\frac{4}{3\sqrt{5}}\right) \approx 53.4°$

(c) To find the perpendicular from P to the line through O and M, express PQ for a general point Q on the line:

$\overrightarrow{PQ} = (\mathbf{i} + \mathbf{j} + \mathbf{2k}) + \mu(\mathbf{3i} + \mathbf{2j} + \mathbf{k})$

Calculate the scalar product of $\overrightarrow{PQ}$ and the direction vector $\mathbf{3i} + \mathbf{2j} + \mathbf{k}$ and equate to zero to find μ.

Solve to find $\mu = -\frac{1}{2}$.

Calculate PQ using μ:

$PQ = \sqrt{5 + 0 + 1.5^2} = \frac{\sqrt{10}}{2}$

Solution to Question 40

June 2023 p33 q8

First, separate the variables:

$\frac{y+4}{y^2+4}\,dy = \frac{1}{x}\,dx$

Integrate both sides:

$\int \frac{y+4}{y^2+4}\,dy = \int \frac{1}{x}\,dx$

This gives:

$\frac{1}{2}\ln(y^2+4) + 2\arctan\frac{y}{2} = \ln x + C$

Use the initial condition $x = 4$ when $y = 2\sqrt{3}$ to find C:

$\frac{1}{2}\ln((2\sqrt{3})^2+4) + 2\arctan\frac{2\sqrt{3}}{2} = \ln 4 + C$

$\frac{1}{2}\ln(16) + 2\times\frac{\pi}{3} = \ln 4 + C$

$\ln 4 + \frac{2\pi}{3} = \ln 4 + C$

$C = \frac{2\pi}{3}$

Substitute $y = 2$ to find x:

$\frac{1}{2}\ln(2^2+4) + 2\arctan\frac{2}{2} = \ln x + \frac{2\pi}{3}$

$\frac{1}{2}\ln 8 + 2\times\frac{\pi}{4} = \ln x + \frac{2\pi}{3}$

$\frac{1}{2}\ln 8 + \frac{\pi}{2} = \ln x + \frac{2\pi}{3}$

$\ln x = \frac{1}{2}\ln 8 + \frac{\pi}{2} - \frac{2\pi}{3}$

$\ln x = \frac{1}{2}\ln 8 - \frac{\pi}{6}$

$x = e^{\frac{1}{2}\ln 8 - \frac{\pi}{6}}$

$x = \frac{\sqrt{8}}{e^{\frac{\pi}{6}}}$

Solution to Question 41

Nov 2022 p33 q10

(a) The rate of water being added is 30 litres per minute, so $a = 30$. The rate of water being lost is proportional to the volume, $0.01V$, so $b = 0.01$.

(b) The differential equation is $\frac{dV}{dt} = 30 - 0.01V$. Separate variables and integrate:

$\int \frac{1}{30-0.01V}\, dV = \int dt$

$-100\ln(30 - 0.01V) = t + C$

Using initial condition $V = 0$ when $t = 0$, find C:

$-100\ln(30) = C$

Thus, $-100\ln(30 - 0.01V) = t - 100\ln(30)$

$100\ln\left(\frac{30}{30-0.01V}\right) = t$

For $V = 1000$:

$100\ln\left(\frac{30}{30-10}\right) = t$

$t = 40.5$

(c) Solving for V in terms of t:

$V = 3000(1 - e^{-0.01t})$

As $t \to \infty$, $e^{-0.01t} \to 0$, so $V \to 3000$.

Solution to Question 42

Nov 2023 p32 q11

(a) Start by separating variables:

$\frac{1}{y^2+y} dy = -\frac{1}{x^2} dx$

Express $\frac{1}{y^2+y}$ in partial fractions:

$\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$

Integrate both sides:

$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int -\frac{1}{x^2} dx$

$\ln|y| - \ln|y+1| = \frac{1}{x} + C$

Use the initial condition $x = 1$, $y = 1$ to find C:

$\ln 1 - \ln 2 = \frac{1}{1} + C$

$-\ln 2 = 1 + C$

$C = -1 - \ln 2$

Substitute back to find y:

$\ln \frac{y}{y+1} = \frac{1}{x} - 1 - \ln 2$

$\frac{y}{y+1} = e^{\frac{1}{x} - 1 - \ln 2}$

$\frac{y}{y+1} = \frac{e^{\frac{1}{x}-1}}{2}$

$y = \frac{e^{\frac{1}{x}-1}}{2 - e^{\frac{1}{x}-1}}$

(b) As x tends to infinity, $\frac{1}{x}$ tends to 0, so:

$y = \frac{e^{0-1}}{2-e^{0-1}} = \frac{e^{-1}}{2-e^{-1}} = \frac{1}{2e-1}$

Solution to Question 43

Nov 2023 p33 q8

First, separate the variables:

$e^{4x}\frac{dy}{dx} = \cos^2 3y$

$\frac{dy}{\cos^2 3y} = e^{-4x} dx$

Integrate both sides:

$\int \sec^2 3y \, dy = \int e^{-4x} \, dx$

$\frac{1}{3}\tan 3y = -\frac{1}{4}e^{-4x} + C$

Use the initial condition $y = 0$ when $x = 2$:

$\frac{1}{3}\tan(0) = -\frac{1}{4}e^{-8} + C$

$0 = -\frac{1}{4}e^{-8} + C$

$C = \frac{1}{4}e^{-8}$

Substitute back to find y:

$\frac{1}{3}\tan 3y = -\frac{1}{4}e^{-4x} + \frac{1}{4}e^{-8}$

$\frac{1}{3}\tan 3y = \frac{1}{4}e^{-8} - \frac{1}{4}e^{-4x}$

$\tan 3y = \frac{3}{4}e^{-8} - \frac{3}{4}e^{-4x}$

$y = \frac{1}{3}\arctan\left(\frac{3}{4}e^{-8} - \frac{3}{4}e^{-4x}\right)$

Solution to Question 44

March 2026 p32 q1

Let

$$u = 2^x.$$

Then

$$2^{x+2} = 4u$$

and

$$2^{2-x} = \frac{4}{u}.$$

So the equation becomes

$$4(4u) - 5\left(\frac{4}{u}\right) = 3.$$

Hence

$$16u - \frac{20}{u} = 3.$$

Multiply by u:

$$16u^2 - 20 = 3u.$$

So

$$16u^2 - 3u - 20 = 0.$$

Using the quadratic formula,

$$u = \frac{3 \pm \sqrt{(-3)^2 - 4(16)(-20)}}{2(16)} = \frac{3 \pm \sqrt{1289}}{32}.$$

Since $u = 2^x > 0$,

$$u = \frac{3 + \sqrt{1289}}{32}.$$

Therefore

$$2^x = \frac{3 + \sqrt{1289}}{32}.$$

Taking logarithms,

$$x = \frac{\ln\left(\frac{3+\sqrt{1289}}{32}\right)}{\ln 2}.$$

$$x = 0.281796\ldots$$

Answer: $x = 0.282$.

Solution to Question 45

March 2026 p32 q2

(a) The condition

$$|z| = 9$$

means the points lie on a circle with centre O and radius 9.

The condition

$$\frac{1}{2}\pi \le \arg z < \pi$$

restricts the locus to the part of the circle in the second quadrant.

So the locus is the arc of the circle $|z| = 9$ from $9i$ to -9, with $9i$ included and -9 not included.

(b) The transformation $z \mapsto z^*$ reflects the locus in the real axis.

The transformation $z^* \mapsto z^* + 3$ then translates the reflected locus 3 units in the positive real direction.

So the locus is an arc of a circle with centre 3 and radius 9, from $3 - 9i$ to -6, with $3 - 9i$ included and -6 not included.

Solution to Question 46

March 2026 p32 q3

Use

$$\sin 2x = 2 \sin x \cos x.$$

Then

$$3 \sin x \sin 2x = 3 \sin x(2 \sin x \cos x) = 6 \sin^2 x \cos x.$$

So

$$\int 3 \sin x \sin 2x \, dx = \int 6 \sin^2 x \cos x \, dx.$$

Let $u = \sin x$, so $du = \cos x \, dx$.

Then

$$\int 6 \sin^2 x \cos x \, dx = \int 6u^2 \, du = 2u^3 = 2 \sin^3 x.$$

Therefore

$$\int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} 3 \sin x \sin 2x \, dxrn = \left[2 \sin^3 x\right]_{\frac{1}{4}\pi}^{\frac{1}{3}\pi}.$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)^3 - 2\left(\frac{\sqrt{2}}{2}\right)^3.$$

$$= \frac{3\sqrt{3}}{4} - \frac{\sqrt{2}}{2}.$$

Answer: $\frac{3}{4}\sqrt{3} - \frac{1}{2}\sqrt{2}$.

Solution to Question 47

March 2026 p32 q4

(a) In the expansion of $(1-ax)^{\frac{2}{5}}$, the coefficient of x^3 is

$$\frac{\frac{2}{5}\left(\frac{2}{5}-1\right)\left(\frac{2}{5}-2\right)}{3!}(-a)^3.$$

So

$$\frac{\frac{2}{5}\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)}{6}(-a)^3 = 1.$$

$$\frac{8}{125}(-a)^3 = 1.$$

$$-a = \frac{5}{2}.$$

Therefore

$$a = -\frac{5}{2}.$$

(b) Since $a = -\frac{5}{2}$,

$$(1-ax)^{\frac{2}{5}} = \left(1+\frac{5}{2}x\right)^{\frac{2}{5}}.$$

The coefficient of x^3 in this expansion is 1.

The coefficient of x^4 in $\left(1+\frac{5}{2}x\right)^{\frac{2}{5}}$ is

$$\frac{\frac{2}{5}\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)}{4!}\left(\frac{5}{2}\right)^4.$$

$$= -\frac{13}{8}.$$

To get the coefficient of x^4 in

$$(2x+1)(1-ax)^{\frac{2}{5}},$$

we need:

$$1 \times \text{coefficient of } x^4$$

and

$$2x \times \text{coefficient of } x^3.$$

Hence the required coefficient is

$$-\frac{13}{8} + 2(1) = \frac{3}{8}.$$

Answer: $\frac{3}{8}$.

(c) The binomial expansion is valid when

$$\left|\frac{5}{2}x\right| < 1.$$

Therefore

$$|x| < \frac{2}{5}.$$

Answer: $-\frac{2}{5} < x < \frac{2}{5}$.

Solution to Question 48

March 2026 p32 q5

(a) Rationalise the denominator:

$$z = \frac{3 + \lambda i}{\lambda + 2i} \times \frac{\lambda - 2i}{\lambda - 2i}.$$

$$z = \frac{(3 + \lambda i)(\lambda - 2i)}{\lambda^2 + 4}.$$

Expanding the numerator,

$$(3 + \lambda i)(\lambda - 2i) = 3\lambda - 6i + \lambda^2 i + 2\lambda.$$

$$= 5\lambda + (\lambda^2 - 6)i.$$

So

$$z = \frac{5\lambda}{\lambda^2 + 4} + \frac{\lambda^2 - 6}{\lambda^2 + 4} i.$$

For $\arg z = \frac{1}{4}\pi$, the real and imaginary parts must be equal and positive.

Therefore

$$5\lambda = \lambda^2 - 6.$$

$$\lambda^2 - 5\lambda - 6 = 0.$$

$$(\lambda - 6)(\lambda + 1) = 0.$$

So

$$\lambda = 6 \quad \text{or} \quad \lambda = -1.$$

Since the real part must be positive, $\lambda = 6$.

Answer: $\lambda = 6$.

(b) When $\lambda = 6$,

$$z = \frac{3 + 6i}{6 + 2i}.$$

Use

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}.$$

So

$$|z| = \frac{|3 + 6i|}{|6 + 2i|}.$$

$$|z| = \frac{\sqrt{3^2 + 6^2}}{\sqrt{6^2 + 2^2}} = \frac{\sqrt{45}}{\sqrt{40}}.$$

$$|z| = \sqrt{\frac{9}{8}} = \frac{3\sqrt{2}}{4}.$$

Answer: $\frac{3\sqrt{2}}{4}$.

Solution to Question 49

March 2026 p32 q6

(a) Since $(x^2 + x + 1)$ is a factor of $p(x)$, divide $p(x)$ by $x^2 + x + 1$.

$$p(x) = 2x^4 + ax^3 + 4x^2 + bx - 3.$$

Using polynomial division, the remainder is

$$(b - 2)x + (a - 7).$$

For $x^2 + x + 1$ to be a factor, the remainder must be zero.

Therefore

$$b - 2 = 0$$

and

$$a - 7 = 0.$$

So

$$a = 7, \quad b = 2.$$

Answer: $a = 7, b = 2$.

(b) With $a = 7$ and $b = 2$,

$$p(x) = 2x^4 + 7x^3 + 4x^2 + 2x - 3.$$

Substitute $x = -3$:

$$p(-3) = 2(-3)^4 + 7(-3)^3 + 4(-3)^2 + 2(-3) - 3.$$

$$= 162 - 189 + 36 - 6 - 3 = 0.$$

Since $p(-3) = 0$, by the factor theorem, $(x + 3)$ is a factor of $p(x)$.

Solution to Question 50

March 2026 p32 q7

(a) Sketch the graphs

$$y = \ln x$$

and

$$y = \operatorname{cosec} \frac{1}{2}x$$

for $0 < x < \pi$.

On this interval, $y = \ln x$ is increasing.

Also, $\sin \frac{1}{2}x$ is increasing on $0 < x < \pi$, so $y = \operatorname{cosec} \frac{1}{2}x$ is decreasing.

Therefore the two graphs can meet at most once.

Since the sketch shows one intersection, the equation has exactly one root in $0 < x < \pi$.

(b) Let

$$f(x) = \ln x - \operatorname{cosec} \frac{1}{2}x.$$

At $x = 2.6$,

$$f(2.6) = \ln(2.6) - \operatorname{cosec}(1.3) = -0.0823\ldots$$

At $x = 2.9$,

$$f(2.9) = \ln(2.9) - \operatorname{cosec}(1.45) = 0.0574\ldots$$

Since there is a change of sign, the root lies between 2.6 and 2.9.

(c) Using

$$x_{n+1} = \exp\left(\operatorname{cosec} \frac{1}{2}x_n\right),$$

and starting with $x_0 = 2.6$:

$$x_1 = 2.82306$$

$$x_2 = 2.75335$$

$$x_3 = 2.77082$$

$$x_4 = 2.76609$$

$$x_5 = 2.76734$$

$$x_6 = 2.76701$$

$$x_7 = 2.76710$$

Therefore the root correct to 3 decimal places is

$$x = 2.767.$$

Answer: 2.767.

Solution to Question 51

March 2026 p32 q8

Start with

$$ye^{3x}\frac{dy}{dx} = x(y+5).$$

Separate the variables:

$$\frac{y}{y+5}\frac{dy}{dx} = xe^{-3x}.$$

So

$$\int \frac{y}{y+5}\,dy = \int xe^{-3x}\,dx.$$

For the left-hand side, write

$$\frac{y}{y+5} = 1 - \frac{5}{y+5}.$$

Hence

$$\int \frac{y}{y+5}\,dy = y - 5\ln(y+5).$$

For the right-hand side, integrate by parts:

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}.$$

Therefore

$$y - 5\ln(y+5) = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.$$

Use $y = 0$ when $x = 0$:

$$0 - 5\ln 5 = -\frac{1}{9} + C.$$

So

$$C = \frac{1}{9} - 5\ln 5.$$

Therefore

$$y - 5\ln(y+5) = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9} - 5\ln 5.$$

Equivalently,

$$y - 5\ln\left(\frac{y+5}{5}\right) = \frac{1}{9} - \left(\frac{x}{3} + \frac{1}{9}\right)e^{-3x}.$$

Solution to Question 52

March 2026 p32 q9

(a) Use

$$x = \sqrt{3}\tan u.$$

Then

$$dx = \sqrt{3}\sec^2 u\, du.$$

Also,

$$3 + x^2 = 3 + 3\tan^2 u = 3\sec^2 u.$$

and

$$x^3 = (\sqrt{3}\tan u)^3 = 3\sqrt{3}\tan^3 u.$$

Therefore

$$\frac{x^3}{3+x^2}\, dxrn = \frac{3\sqrt{3}\tan^3 u}{3\sec^2 u}\cdot\sqrt{3}\sec^2 u\, du.$$
$$= 3\tan^3 u\, du.$$

When $x = 1$,

$$1 = \sqrt{3}\tan u$$

so

$$\tan u = \frac{1}{\sqrt{3}}$$

and

$$u = \frac{\pi}{6}.$$

When $x = 3$,

$$3 = \sqrt{3}\tan u$$

so

$$\tan u = \sqrt{3}$$

and

$$u = \frac{\pi}{3}.$$

Hence

$$I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 3\tan^3 u\, du.$$

(b) It is easier to divide first:

$$\frac{x^3}{3+x^2} = x - \frac{3x}{x^2+3}.$$

So

$$I = \int_1^3 \left(x - \frac{3x}{x^2+3}\right) dx.$$

$$I = \left[\frac{x^2}{2} - \frac{3}{2}\ln(x^2+3)\right]_1^3.$$

$$I = \left(\frac{9}{2} - \frac{3}{2}\ln 12\right) - \left(\frac{1}{2} - \frac{3}{2}\ln 4\right).$$

$$I = 4 - \frac{3}{2}\ln 3.$$

Answer: $4 - \frac{3}{2}\ln 3$.

Solution to Question 53

March 2026 p32 q10

(a) Start with

$$y^2 = k\frac{x-2}{x+2}.$$

Differentiate implicitly:

$$2y\frac{dy}{dx} = rnk\frac{(x+2)-(x-2)}{(x+2)^2}.$$

So

$$2y\frac{dy}{dx} = \frac{4k}{(x+2)^2}.$$

From the original equation,

$$k = \frac{y^2(x+2)}{x-2}.$$

Substitute this into the derivative equation:

$$2y\frac{dy}{dx} = \frac{4}{(x+2)^2} \cdot \frac{y^2(x+2)}{x-2}.$$

$$2y\frac{dy}{dx} = \frac{4y^2}{(x+2)(x-2)}.$$

$$2y\frac{dy}{dx} = \frac{4y^2}{x^2-4}.$$

Therefore

$$\frac{dy}{dx} = \frac{2y}{x^2-4}.$$

(b) When $k = 5$ and $x = 3$,

$$y^2 = 5\frac{3-2}{3+2} = 1.$$

So

$$y = 1 \quad \text{or} \quad y = -1.$$

Using

$$\frac{dy}{dx} = \frac{2y}{x^2-4},$$

at $x = 3$:

$$\frac{dy}{dx} = \frac{2y}{9-4} = \frac{2y}{5}.$$

So the two gradients are

$$m_1 = \frac{2}{5}, \quad m_2 = -\frac{2}{5}.$$

The tangents make angles

$$\tan^{-1}\left(\frac{2}{5}\right)$$

and

$$-\tan^{-1}\left(\frac{2}{5}\right)$$

with the positive x-axis.

Therefore the angle between the tangents is

$$2\tan^{-1}\left(\frac{2}{5}\right).$$

Answer: $2\tan^{-1}\left(\frac{2}{5}\right)$.

Solution to Question 54

March 2026 p32 q11

(a) Write

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}.$$

The perpendicular distance from A to the line through O and B is

$$\frac{|\overrightarrow{OA} \times \overrightarrow{OB}|}{|\overrightarrow{OB}|}.$$

Now

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}.$$

$$= \begin{pmatrix} 10 \\ -12 \\ -4 \end{pmatrix}.$$

So

$$|\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{10^2 + (-12)^2 + (-4)^2} = \sqrt{260} = 2\sqrt{65}.$$

Also,

$$|\overrightarrow{OB}| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6.$$

Hence the perpendicular distance is

$$\frac{2\sqrt{65}}{6} = \frac{1}{3}\sqrt{65}.$$

(b) We have

$$\overrightarrow{OC} = \begin{pmatrix} 3 \\ p \\ q \end{pmatrix}.$$

Also

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}.$$

Since $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$,

$$\overrightarrow{OC} \cdot \overrightarrow{AB} = 0.$$

So

$$\begin{pmatrix}3\\p\\q\end{pmatrix}\cdot\begin{pmatrix}2\\0\\5\end{pmatrix}=0.$$

$$6+5q=0.$$

Therefore

$$q=-\frac{6}{5}.$$

Since angle $AOC=$ angle COB,

$$\frac{\overrightarrow{OA}\cdot\overrightarrow{OC}}{|\overrightarrow{OA}|\,|\overrightarrow{OC}|}=\frac{\overrightarrow{OC}\cdot\overrightarrow{OB}}{|\overrightarrow{OC}|\,|\overrightarrow{OB}|}.$$

Cancel $|\overrightarrow{OC}|$:

$$\frac{\overrightarrow{OA}\cdot\overrightarrow{OC}}{|\overrightarrow{OA}|}=\frac{\overrightarrow{OC}\cdot\overrightarrow{OB}}{|\overrightarrow{OB}|}.$$

Now

$$|\overrightarrow{OA}|=\sqrt{2^2+2^2+(-1)^2}=3$$

and

$$|\overrightarrow{OB}|=6.$$

So

$$2(\overrightarrow{OA}\cdot\overrightarrow{OC})=\overrightarrow{OC}\cdot\overrightarrow{OB}.$$

Now

$$\overrightarrow{OA}\cdot\overrightarrow{OC}=2(3)+2p+(-1)q=6+2p-q.$$

Using $q=-\frac{6}{5}$,

$$\overrightarrow{OA}\cdot\overrightarrow{OC}=6+2p+\frac{6}{5}=\frac{36}{5}+2p.$$

Also

$$\overrightarrow{OC}\cdot\overrightarrow{OB}=3(4)+2p+4q=12+2p+4q.$$

Using $q=-\frac{6}{5}$,

$$\overrightarrow{OC}\cdot\overrightarrow{OB}=12+2p-\frac{24}{5}=\frac{36}{5}+2p.$$

Therefore

$$2\left(\frac{36}{5}+2p\right)=\frac{36}{5}+2p.$$

$$\frac{72}{5}+4p=\frac{36}{5}+2p.$$

$$2p=-\frac{36}{5}.$$

So

$$p=-\frac{18}{5}.$$

Answer: $p=-\frac{18}{5}$, $q=-\frac{6}{5}$.