
CAMBRIDGE INTERNATIONAL AS & A LEVEL MATHEMATICS 9709

Cambridge 9709 Paper 1 Final Revision Booklet

10-Day Exam Preparation

Concise theory, curated exam-style practice and fully worked solutions for the final stretch before Paper 1.

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How to Use This Booklet

This booklet is designed for the final 10 days before the exam. Read each theory recap quickly, then attempt the selected questions without looking at the solutions.

Mark every weak area immediately. If a question exposes a method you cannot complete, revise that small method first, then redo the question from a blank page.

In the final days, move from topic practice to mixed practice. Paper 1 rewards quick recognition, accurate algebra and clean working, so train those habits every day.

10-Day Revision Plan

Day 1	Quadratics + Functions Refresh completing the square, inverse functions, domains and composites.
Day 2	Coordinate Geometry Straight lines, gradients, circles and intersection algebra.
Day 3	Circular Measure + Trigonometry Basics Radians, sectors, exact trig values and general angles.
Day 4	Trigonometric Graphs, Identities, Equations Graphs, inverse trig, identities and full-solution intervals.
Day 5	Series + Binomial Expansion AP/GP formulae, finite and infinite sums, coefficients and constants.
Day 6	Differentiation Basics Differentiate, tangents, increasing/decreasing intervals.
Day 7	Differentiation Applications Stationary points, rates of change and modelling questions.
Day 8	Integration Curve equations, areas, volumes and constants of integration.
Day 9	Mixed Topic Practice Attempt mixed questions under timed conditions; mark weak methods.
Day 10	Full Final Review Redo mistakes, memorise formulas, complete one final exam-style set.

Quadratics

Task Types

- Solving quadratic equations by factorisation
- Completing the square
- Solving simultaneous equations: one linear and one quadratic
- Solving more complex quadratic equations
- Solving quadratic inequalities
- The number of roots of a quadratic equation
- Intersection of a line and a quadratic curve

Essential Reminders

- Use factorisation, completing the square or the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- For $ax^2 + bx + c$, the discriminant $\Delta = b^2 - 4ac$: two roots if $\Delta > 0$, one repeated root if $\Delta = 0$, no real roots if $\Delta < 0$.
- Line/curve intersections usually reduce to one quadratic equation. A tangent means the quadratic has a repeated root.
- For inequalities, find critical roots first, then use a sign diagram or the shape of the parabola.

Common Traps

- Forgetting to reverse logic when the coefficient of x^2 is negative.
- Writing $k < 3$ when the boundary value $k = 3$ is allowed for a restricted quadratic.
- Losing one solution after squaring or after substituting $u = x^2$, \sqrt{x} , or e^x -style expressions.

Selected Practice

Question 1

Core

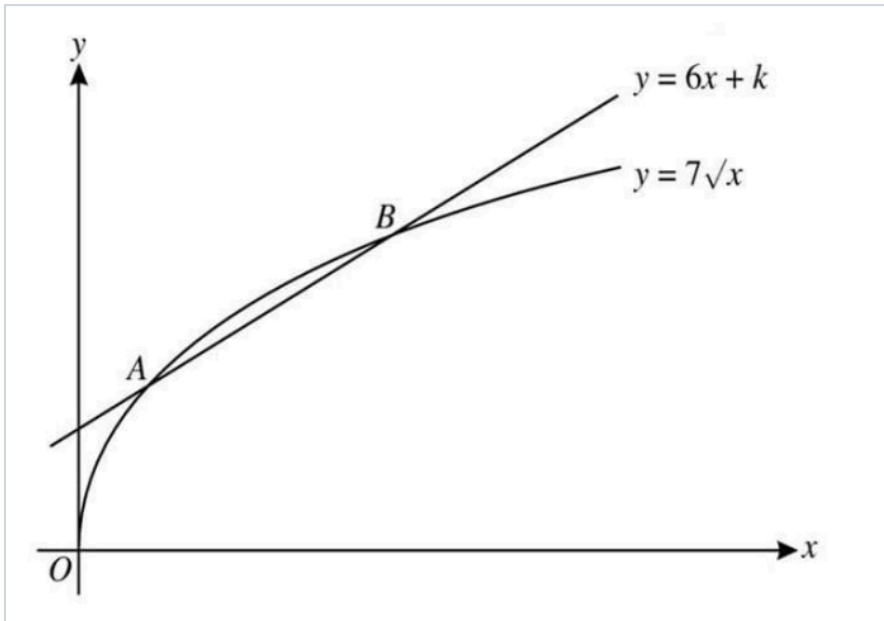
June 2023 p12 q3a · Source ID 297

Express $4x^2 - 24x + p$ in the form $a(x + b)^2 + c$, where a and b are integers and c is to be given in terms of the constant p .

Question 2

Exam

June 2012 P11 Q5 · Source ID 377



The diagram shows the curve $y = 7\sqrt{x}$ and the line $y = 6x + k$, where k is a constant. The curve and the line intersect at the points A and B .

For the case where $k = 2$, find the x -coordinates of A and B .

Question 3

Harder

Nov 2022 p11 q1 · Source ID 359

Solve the equation $8x^6 + 215x^3 - 27 = 0$.

Question 4

Core

October 2023 P1 Q1 · Source ID 366

The function f is defined by $f(x) = x^2 - 4x + 8$ for $x \in \mathbb{R}$. Find the set of values of x for which $f(x) < 9$, giving your answer in exact form.

Question 5

Exam

Nov 2023 p11 q2 · Source ID 306

A line has equation $y = 2x + 3$ and a curve has equation $y = cx^2 + 3x - c$, where c is a constant.

Showing all necessary working, determine which of the following statements is correct.

- A. The line and curve intersect only for a particular set of values of c .
- B. The line and curve intersect for all values of c .
- C. The line and curve do not intersect for any values of c .

Question 6

Core

Original bridge question: factorising quadratics · Original

Solve the equation $2x^2 - 7x - 15 = 0$, giving your answers in exact form.

Question 7

Core

Original bridge question: line and quadratic intersection · Original

The line $y = 2x + k$ is a tangent to the curve $y = x^2 - 4x + 7$. Find the possible value of k and the coordinates of the point of contact.

Functions

Task Types

- Inverse function
- Inverse functions domain
- Composite functions
- Discriminant of a composite
- The graph of a function and its inverse
- Transformations of functions
- Stretches
- Combined transformations

Essential Reminders

- To find f^{-1} , write $y = f(x)$, solve for x , then swap x and y .
- A function has an inverse only on a one-one domain. For quadratics, restrict to one side of the vertex.
- Composition means order matters: $fg(x) = f(g(x))$. Check the domain of the inner function first.
- Graphs of f and f^{-1} are reflections in the line $y = x$.

Common Traps

- Choosing the wrong square-root branch when finding an inverse.
- Applying transformations in the wrong order, especially horizontal stretches and translations.
- Not stating the domain or range when the question asks for an inverse.

Selected Practice

Question 8

Exam

Feb/Mar 2023 p12 q9 · Source ID 736

The function f is defined by $f(x) = -3x^2 + 2$ for $x \leq -1$.

- (a) State the range of f .
- (b) Find an expression for $f^{-1}(x)$.

Question 9

Harder

Nov 2023 p13 q7 · Source ID 667

The function f is defined by $f(x) = 1 + \frac{3}{x-2}$ for $x > 2$.

The function g is defined by $g(x) = 2x - 2$ for $x > 0$.

Obtain a simplified expression for $gf(x)$.

Question 10

Core

Nov 2023 p13 q7 · Source ID 745

The function f is defined by $f(x) = 1 + \frac{3}{x-2}$ for $x > 2$.

- (a) State the range of f .
- (b) Obtain an expression for $f^{-1}(x)$ and state the domain of f^{-1} .

Question 11

Exam

June 2020 p13 q9 · Source ID 775

The functions f and g are defined by

$$f(x) = x^2 - 4x + 3 \text{ for } x > c, \text{ where } c \text{ is a constant,}$$

$$g(x) = \frac{1}{x+1} \text{ for } x > -1.$$

- (a) Express $f(x)$ in the form $(x - a)^2 + b$.

It is given that f is a one-one function.

- (b) State the smallest possible value of c .

It is now given that $c = 5$.

- (c) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1} .
- (d) Find an expression for $gf(x)$ and state the range of gf .

Question 12

Harder

Nov 2019 p12 q9 · Source ID 711

Functions f and g are defined by

$$f(x) = 2x^2 + 8x + 1 \text{ for } x \in \mathbb{R},$$

$$g(x) = 2x - k \text{ for } x \in \mathbb{R},$$

where k is a constant.

Find the value of k for which the line $y = g(x)$ is a tangent to the curve $y = f(x)$.

Question 13

Core

Nov 2023 p12 q6 · Source ID 648

The equation of a curve is $y = x^2 - 8x + 5$.

- (a) Find the coordinates of the minimum point of the curve.

The curve is stretched by a factor of 2 parallel to the y -axis and then translated by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

- (b) Find the coordinates of the minimum point of the transformed curve.
- (c) Find the equation of the transformed curve. Give the answer in the form $y = ax^2 + bx + c$, where a , b and c are integers to be found.

Question 14

Core

Original bridge question: transformations of functions · Original

The graph of $y = f(x)$ is transformed to the graph of $y = 2f(x - 3) - 1$. Describe the transformations fully and in the correct order.

Coordinate Geometry

Task Types

- Length of a line segment and midpoint
- Parallel and perpendicular lines
- Equations of straight lines
- The equation of a circle
- Problems involving intersections of lines and circles
- Coordinate Geometry Review 3

Essential Reminders

- Distance: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Midpoint: $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.
- Parallel lines have equal gradients; perpendicular gradients multiply to -1 .
- Circle form: $(x - a)^2 + (y - b)^2 = r^2$, centre (a, b) , radius r .
- Intersections are found by substitution, usually leading to a quadratic.

Common Traps

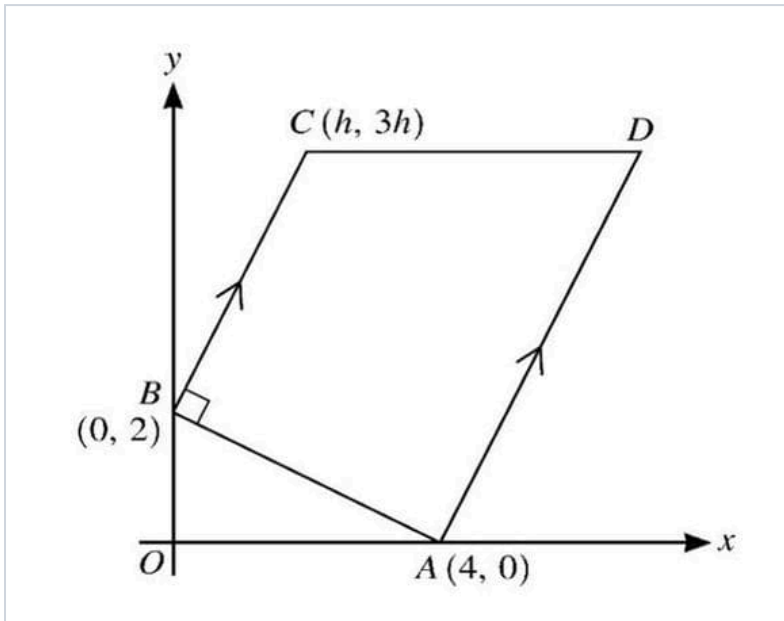
- Using m instead of $-1/m$ for a perpendicular line.
- Not completing the square carefully when finding a circle centre.
- Leaving both intersection coordinates incomplete after solving only for x .

Selected Practice

Question 15

Harder

June 2019 p11 q4 · Source ID 599



The diagram shows a trapezium ABCD in which the coordinates of A, B, and C are $(4, 0)$, $(0, 2)$, and $(h, 3h)$ respectively. The lines BC and AD are parallel, angle $\angle ABC = 90^\circ$ and CD is parallel to the x-axis.

- Find, by calculation, the value of h .
- Hence find the coordinates of D.

Question 16

Core

Nov 2023 p13 q2 · Source ID 620

The circle with equation $(x - 3)^2 + (y - 5)^2 = 40$ intersects the y-axis at points A and B.

- Find the y-coordinates of A and B, expressing your answers in terms of surds.
- Find the equation of the circle which has AB as its diameter.

Question 17

Exam

Nov 2023 p13 q6 · Source ID 117

The curve C_1 has the equation $y = x^2 - 4x + 7$. The curve C_2 has the equation $y^2 = 4x + k$, where k is a constant. The tangent to C_1 at the point where $x = 3$ is also the tangent to C_2 at the point P. Find the value of k and the coordinates of P.

Question 18

Core

Original bridge question: distance and midpoint · Original

The points $A(-2, 5)$ and $B(6, -1)$ are the endpoints of a diameter of a circle. Find the centre and radius of the circle.

Question 19

Core

Original bridge question: parallel and perpendicular lines · Original

The line l has equation $3x - 2y + 8 = 0$. Find the equation of the line through $(4, -1)$ which is perpendicular to l , giving your answer in the form $ax + by + c = 0$.

Question 20

Core

Original bridge question: line and circle intersections · Original

The line $y = x + 1$ intersects the circle $x^2 + y^2 = 25$ at the points A and B . Find the coordinates of A and B .

Circular Measure

Task Types

- Radians
- Arc length and sector area
- Circular measure review

Essential Reminders

- Convert degrees to radians using $\theta_{\text{rad}} = \theta^{\circ} \frac{\pi}{180}$.
- Arc length $s = r\theta$, sector area $A = \frac{1}{2}r^2\theta$, with θ in radians.
- Segment area often means sector area minus triangle area.

Common Traps

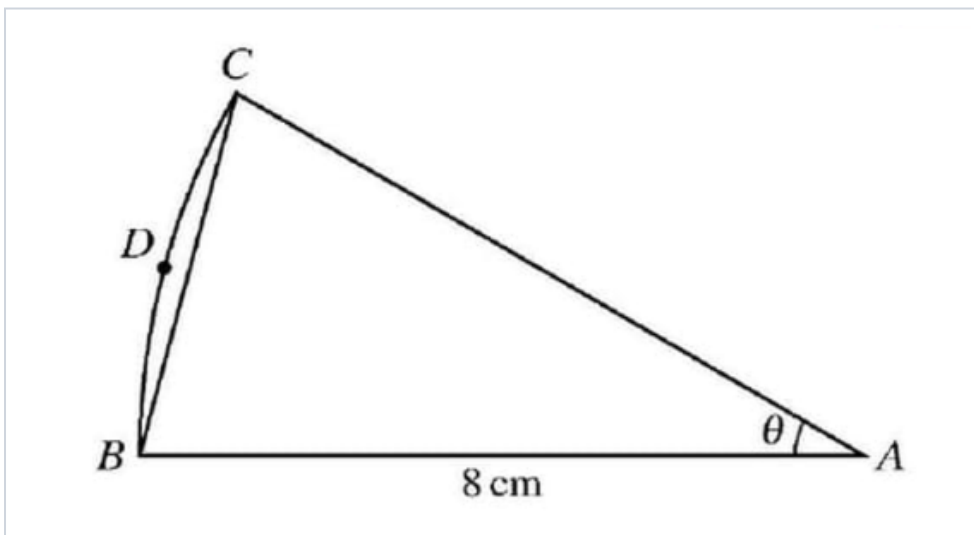
- Using degree mode when the formula requires radians.
- Mixing up perimeter of a sector with arc length only.
- Subtracting the wrong triangle when finding a shaded segment.

Selected Practice

Question 21

Harder

Exam-style question · Source ID 242



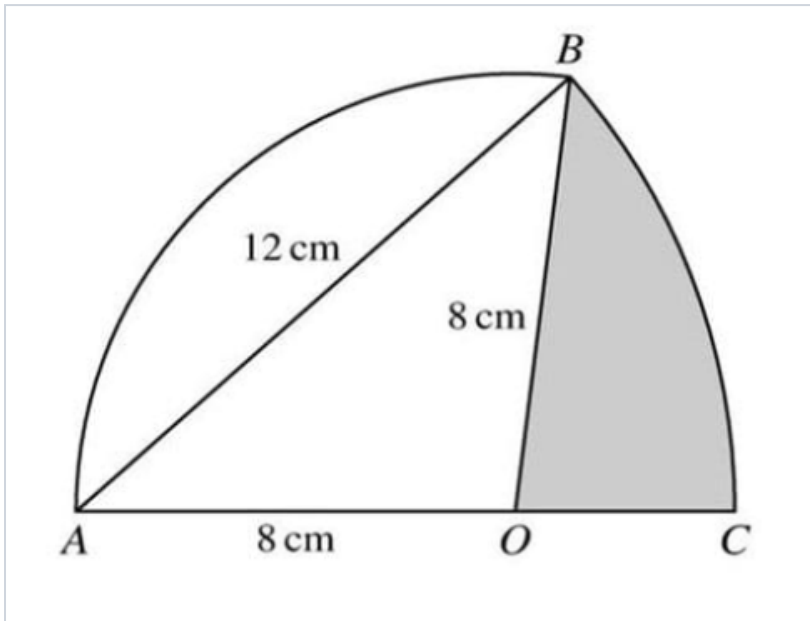
The diagram shows a sector ABC of a circle with centre A and radius 8 cm . The area of the sector is $\frac{16}{3}\pi \text{ cm}^2$. The point D lies on the arc BC .

Find the perimeter of the segment BCD .

Question 22

Core

Nov 2020 p13 q9 · Source ID 201



In the diagram, arc AB is part of a circle with centre O and radius 8 cm . Arc BC is part of a circle with centre A and radius 12 cm , where AOC is a straight line.

- Find angle BAO in radians.
- Find the area of the shaded region.
- Find the perimeter of the shaded region.

Question 23

Core

Original bridge question: radians · Original

Convert 135° to radians, giving your answer as an exact multiple of π . Hence find the length of an arc of radius 8 cm which subtends an angle of 135° at the centre.

Trigonometry

Task Types

- Angles between 0 degrees and 90 degrees
- Trigonometric ratios of general angles
- Graphs of trigonometric functions
- Inverse trigonometric functions
- Trigonometric equations
- Trigonometric identities
- Further trigonometric equations: linear
- Further trigonometric equations: quadratic
- Trigonometry review

Essential Reminders

- Know exact values for 0° , 30° , 45° , 60° , 90° .
- Use CAST or the unit circle to place all solutions in the required interval.
- Start identity proofs from the more complicated side; change everything to sin and cos when stuck.
- For $a \sin x + b \cos x$, use identities or rearrange carefully; never divide by a trig expression without checking zero cases.

Common Traps

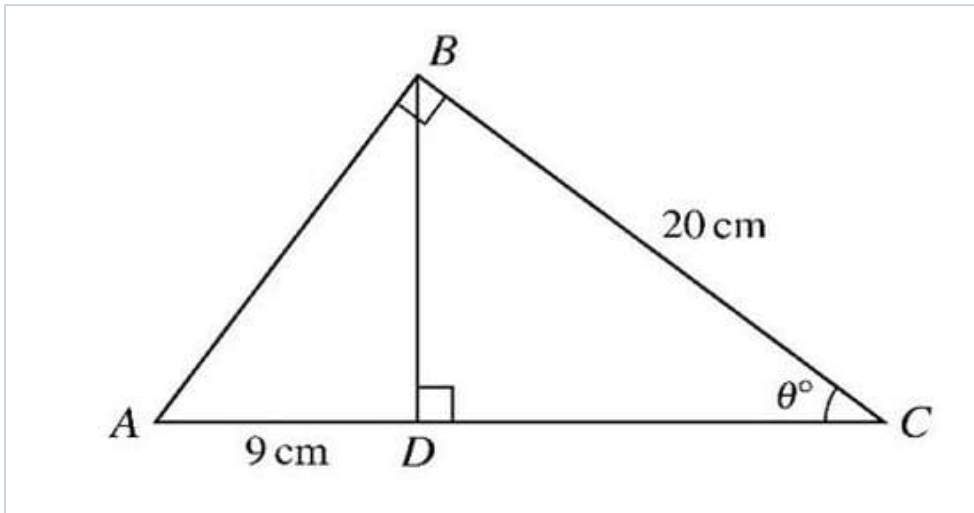
- Giving only the principal calculator solution.
- Mixing degrees and radians in one question.
- Losing solutions after dividing by $\sin x$, $\cos x$, or $\tan x$.

Selected Practice

Question 24

Harder

Nov 2018 p12 q6 · Source ID 560



The diagram shows a triangle ABC in which $BC = 20\text{ cm}$ and angle ABC is 90° . The perpendicular from B to AC meets AC at D and $AD = 9\text{ cm}$. Angle BCA is θ° .

- By expressing the length of BD in terms of θ in each of the triangles ABD and DBC , show that $20 \sin^2 \theta = 9 \cos \theta$.
- Hence, showing all necessary working, calculate θ .

Question 25

Core

March 2019 p12 q7a · Source ID 442

Solve the equation $3 \sin^2 2\theta + 8 \cos 2\theta = 0$ for $0^\circ \leq \theta \leq 180^\circ$.

Question 26

Exam

November 2020 P12 Q11 · Source ID 525

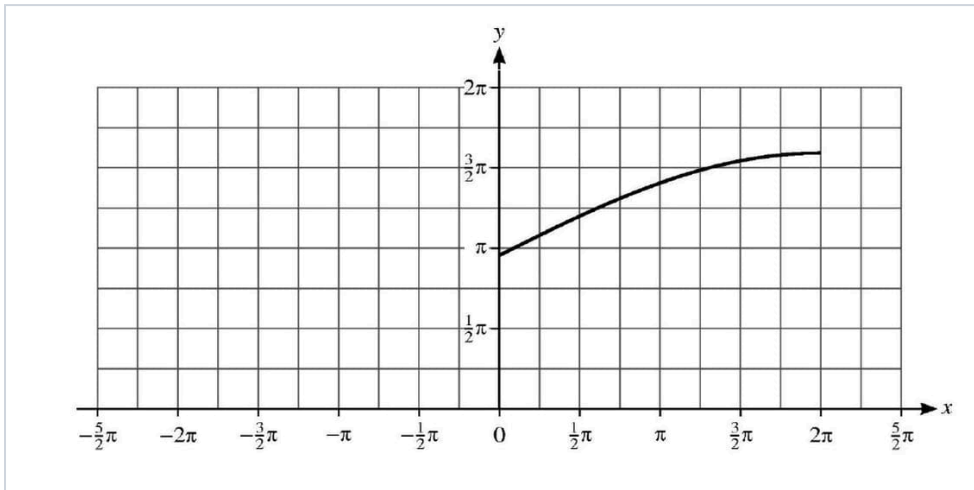
A curve has equation $y = 3 \cos 2x + 2$ for $0 \leq x \leq \pi$.

- State the greatest and least values of y .
- Sketch the graph of $y = 3 \cos 2x + 2$ for $0 \leq x \leq \pi$.
- By considering the straight line $y = kx$, where k is a constant, state the number of solutions of the equation $3 \cos 2x + 2 = kx$ for $0 \leq x \leq \pi$ in each of the following cases.
 - $k = -3$
 - $k = 1$
 - $k = 3$

Question 27

Harder

Exam-style question · Source ID 538



The diagram shows the graph of $y = f(x)$ where the function f is defined by

$$f(x) = 3 + 2 \sin \frac{1}{4}x \text{ for } 0 \leq x \leq 2\pi.$$

(a) On the diagram above, sketch the graph of $y = f^{-1}(x)$. [2]

(b) Find an expression for $f^{-1}(x)$. [2]

(c) The diagram above shows part of the graph of the function $g(x) = 3 + 2 \sin \frac{1}{4}x$ for $-2\pi \leq x \leq 2\pi$.

Complete the sketch of the graph of $g(x)$ on the diagram above and hence explain whether the function g has an inverse. [2]

Question 28

Core

Exam-style question · Source ID 482

Prove the identity $\left(\frac{1}{\cos x} - \tan x\right) \left(\frac{1}{\sin x} + 1\right) \equiv \frac{1}{\tan x}$.

Question 29

Exam

Exam-style question · Source ID 390

(a) Prove the identity $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} \equiv \frac{2}{\cos\theta}$.

(b) Hence solve the equation $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{3}{\sin\theta}$, for $0 \leq \theta \leq 2\pi$.

Question 30

Harder

Nov 2022 p13 q6 · Source ID 577

It is given that $\alpha = \cos^{-1}\left(\frac{8}{17}\right)$.

Find, without using the trigonometric functions on your calculator, the exact value of $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha}$.

Question 31

Core

Original bridge question: general angles · Original

Given that $\sin\theta = -\frac{1}{2}$ and $0^\circ \leq \theta \leq 360^\circ$, find the possible values of θ .

Question 32

Core

Original bridge question: trigonometric equations · Original

Solve $2 \cos x - 1 = 0$ for $0 \leq x \leq 2\pi$.

Series

Task Types

- Arithmetic progression
- Geometric progression
- Trigonometric arithmetic progression
- Trigonometric geometric progression
- Series review

Essential Reminders

- AP: $u_n = a + (n - 1)d$, $S_n = \frac{n}{2}(2a + (n - 1)d) = \frac{n}{2}(a + l)$.
- GP: $u_n = ar^{n-1}$, $S_n = \frac{a(1-r^n)}{1-r}$, and $S_\infty = \frac{a}{1-r}$ for $|r| < 1$.
- When trig terms form a progression, use exact trig values or identities before applying AP/GP formulae.

Common Traps

- Using n instead of $n - 1$ in the n th term.
- Applying S_∞ when $|r| \geq 1$.
- Rounding too early in a progression question.

Selected Practice

Question 33

Harder

June 2023 p13 q8 · Source ID 822

A progression has first term a and second term $\frac{a^2}{a+2}$, where a is a positive constant.

For the case where the progression is arithmetic and $a = 6$, determine the least value of n required for the sum of the first n terms to be less than -480 .

Question 34

Core

June 2021 p11 q5 · Source ID 881

The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively.

Given that k is negative, find the sum to infinity of the progression.

Question 35

Exam

Nov 2021 p12 q5 · Source ID 944

The first, third and fifth terms of an arithmetic progression are $2 \cos x$, $-6\sqrt{3} \sin x$ and $10 \cos x$ respectively, where $\frac{1}{2}\pi < x < \pi$.

(a) Find the exact value of x . [3]

(b) Hence find the exact sum of the first 25 terms of the progression. [3]

Question 36

Harder

Nov 2023 p12 q5 · Source ID 950

The first, second and third terms of a geometric progression are $\sin \theta$, $\cos \theta$ and $2 - \sin \theta$ respectively, where θ radians is an acute angle.

(a) Find the value of θ .

(b) Using this value of θ , find the sum of the first 10 terms of the progression. Give the answer in the form $\frac{b}{\sqrt{c-1}}$, where b and c are integers to be found.

Question 37

Core

June 2023 p11 q6 · Source ID 796

The first three terms of an arithmetic progression are $\frac{p^2}{6}$, $2p - 6$ and p .

(a) Given that the common difference of the progression is not zero, find the value of p .

(b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p - 6$.

Binomial Expansion

Task Types

- Expansion of three terms
- Finding the coefficient of a given term
- Finding the value of the constant
- Binomial expansion review

Essential Reminders

- $(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$ for positive integer n .
- For a specific power of x , write the general term and match exponents.
- With three terms, expand in stages or group two terms first.

Common Traps

- Using $\binom{n}{r}$ with the wrong value of r .
- Forgetting signs when the bracket contains $-x$.
- Expanding more terms than needed for a coefficient question.

Selected Practice

Question 38

Exam

Nov 2023 p11 q1 · Source ID 955

- (a) Expand $(1 + 3x)^6$ in ascending powers of x up to, and including, the term in x^2 .
- (b) Hence find the coefficient of x^2 in the expansion of $(1 - 7x + x^2)(1 + 3x)^6$.

Question 39

Harder

June 2020 p12 q1 · Source ID 982

- (a) Find the coefficient of x^2 in the expansion of $(x - \frac{2}{x})^6$.
- (b) Find the coefficient of x^2 in the expansion of $(2 + 3x^2)(x - \frac{2}{x})^6$.

Question 40

Core

Nov 2023 p13 q4 · Source ID 1004

- (a) Expand the following in ascending powers of x up to and including the term in x^2 .
- (i) $(1 + 2x)^5$.
- (ii) $(1 - ax)^6$, where a is a constant.

In the expansion of $(1 + 2x)^5(1 - ax)^6$, the coefficient of x^2 is -5 .

- (b) Find the possible values of a .

Question 41

Exam

Feb/Mar 2022 p12 q3 · Source ID 1036

Find the term independent of x in each of the following expansions.

(a) $\left(3x + \frac{2}{x^2}\right)^6$

(b) $\left(3x + \frac{2}{x^2}\right)^6(1 - x^3)$

Differentiation

Task Types

- Differentiate
- Increasing and decreasing functions
- Equation of a tangent
- Stationary points and nature
- Rate of change
- Application

Essential Reminders

- If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$.
- At stationary points, $\frac{dy}{dx} = 0$. Use a sign test or $\frac{d^2y}{dx^2}$ for nature.
- A tangent gradient is $\frac{dy}{dx}$ at the point; a normal gradient is the negative reciprocal.
- Rate of change questions require units and the correct variable with respect to time.

Common Traps

- Substituting into y before finding the gradient.
- Confusing tangent and normal gradients.
- Classifying a stationary point without evidence.

Selected Practice

Question 42

Harder

June 2011 p13 q4 · Source ID 1183

Differentiate $\frac{2x^3+5}{x}$ with respect to x .

Question 43

Core

Nov 2022 p11 q8 · Source ID 1136

The function f is defined by $f(x) = 2 - \frac{3}{4x-p}$ for $x > \frac{p}{4}$, where p is a constant.Find $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither.

Question 44

Exam

June 2023 p12 q11 · Source ID 1161

The equation of a curve is

$$y = k\sqrt{4x + 1} - x + 5,$$

where k is a positive constant.

- (a) Find $\frac{dy}{dx}$.
- (b) Find the x -coordinate of the stationary point in terms of k .
- (c) Given that $k = 10.5$, find the equation of the normal to the curve at the point where the tangent to the curve makes an angle of $\arctan(2)$ with the positive x -axis.

Question 45

Harder

Nov 2023 p12 q10 · Source ID 1055

The equation of a curve is $y = f(x)$, where $f(x) = (4x - 3)^{\frac{5}{3}} - \frac{20}{3}x$.

- (a) Find the x -coordinates of the stationary points of the curve and determine their nature.
- (b) State the set of values for which the function f is increasing.

Question 46

Core

Nov 2023 p13 q9 · Source ID 1099

A curve has equation $y = 2x^{\frac{1}{2}} - 1$.

- (a) Find the equation of the normal to the curve at the point $A(4, 3)$, giving your answer in the form $y = mx + c$.

A point is moving along the curve $y = 2x^{\frac{1}{2}} - 1$ in such a way that at A the rate of increase of the x -coordinate is 3 cm s^{-1} .

- (b) Find the rate of increase of the y -coordinate at A .

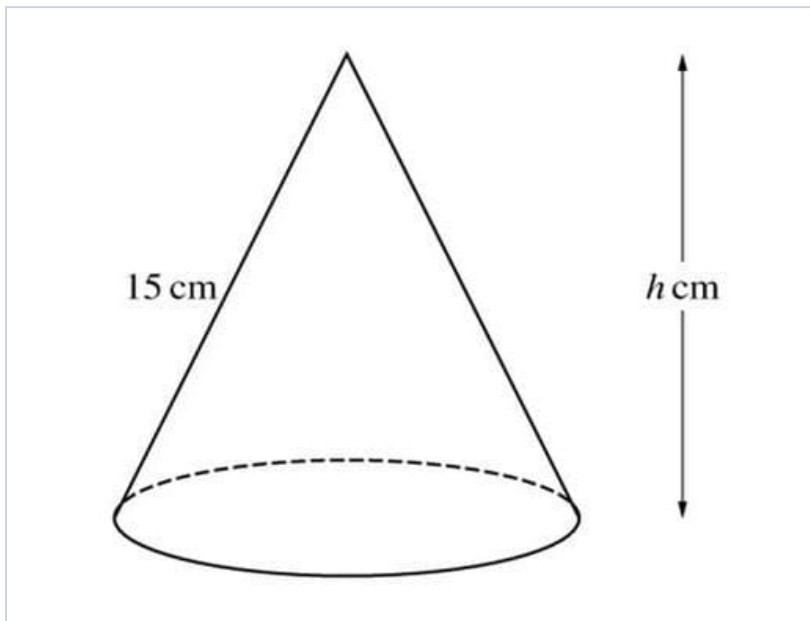
At A the moving point suddenly changes direction and speed, and moves down the normal in such a way that the rate of decrease of the y -coordinate is constant at 5 cm s^{-1} .

- (c) As the point moves down the normal, find the rate of change of its x -coordinate.

Question 47

Exam

Nov 2019 p12 q5 · Source ID 1185



The diagram shows a solid cone which has a slant height of 15 cm and a vertical height of h cm.

(i) Show that the volume, V cm³, of the cone is given by $V = \frac{1}{3}\pi(225h - h^3)$.

[The volume of a cone of radius r and vertical height h is $\frac{1}{3}\pi r^2 h$.]

(ii) Given that h can vary, find the value of h for which V has a stationary value. Determine, showing all necessary working, the nature of this stationary value.

Integration

Task Types

- Integration
- Equation of a curve
- Finding area
- Finding volume
- Cross-topic review

Essential Reminders

- If $\frac{dy}{dx} = ax^n$, then $y = \frac{a}{n+1}x^{n+1} + C$, for $n \neq -1$.
- Use a given point to find C .
- Area under a curve is $\int_a^b y dx$; area between curves is upper minus lower.
- Volume of revolution about the x -axis is $V = \pi \int_a^b y^2 dx$.

Common Traps

- Forgetting the constant of integration.
- Using signed area when the question asks for total area.
- Missing the factor π in volume questions.

Selected Practice

Question 48

Harder

June 2022 p13 q10 · Source ID 1207

The function f is defined by $f(x) = (4x + 2)^{-2}$ for $x > -\frac{1}{2}$.

Find $\int_1^\infty f(x) dx$.

Question 49

Core

Nov 2023 p13 q1 · Source ID 1217

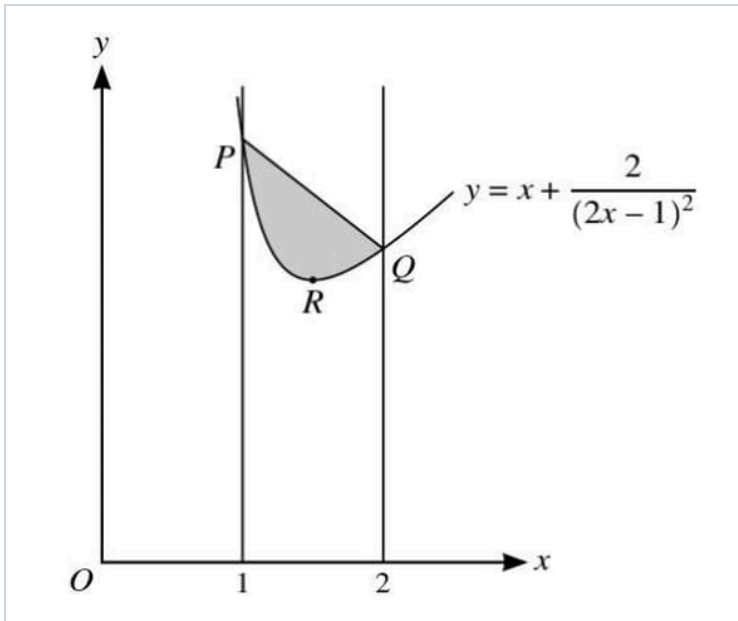
A curve is such that its gradient at a point (x, y) is given by $\frac{dy}{dx} = x - 3x^{-\frac{1}{2}}$. It is given that the curve passes through the point $(4, 1)$.

Find the equation of the curve.

Question 50

Exam

Nov 2023 p13 q11 · Source ID 1260



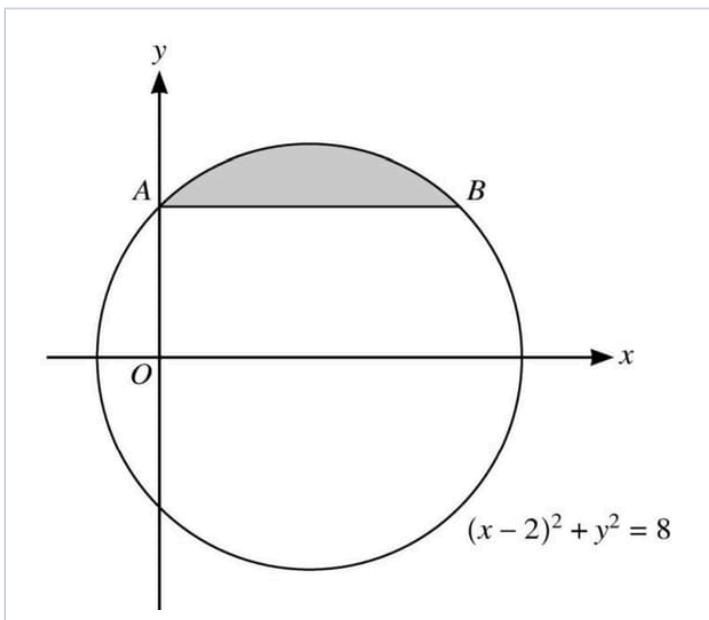
The diagram shows part of the curve with equation $y = x + \frac{2}{(2x-1)^2}$. The lines $x = 1$ and $x = 2$ intersect the curve at P and Q respectively and R is the stationary point on the curve.

- (a) Verify that the x -coordinate of R is $\frac{3}{2}$ and find the y -coordinate of R . [4]
 (b) Find the exact value of the area of the shaded region. [6]

Question 51

Harder

March 2022 p12 q8 · Source ID 1352



The diagram shows the circle with equation $(x-2)^2 + y^2 = 8$. The chord AB of the circle intersects the positive y -axis at A and is parallel to the x -axis.

- (a) Find, by calculation, the coordinates of A and B .
 (b) Find the volume of revolution when the shaded segment, bounded by the circle and the chord AB , is rotated through 360° about the x -axis.

Question 52

Core

Nov 2021 p13 q10 · Source ID 1369

A curve has equation $y = f(x)$ and it is given that

$$f'(x) = \left(\frac{1}{2}x + k\right)^{-2} - (1 + k)^{-2},$$

where k is a constant. The curve has a minimum point at $x = 2$.

(a) Find $f''(x)$ in terms of k and x , and hence find the set of possible values of k .

It is now given that $k = -3$ and the minimum point is at $(2, 3\frac{1}{2})$.

(b) Find $f(x)$.

(c) Find the coordinates of the other stationary point and determine its nature.

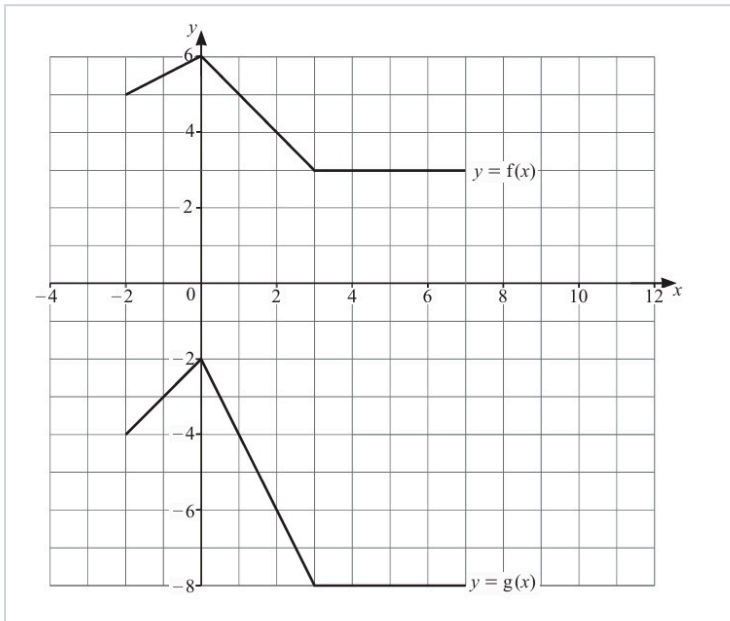
Mixed Exam Practice

Attempt these in exam conditions. Do not group them by topic while working; the purpose is to practise recognising the method quickly.

Question 53

Mixed

June 2025 p12 q1 · Source ID 4096



The diagram shows the graphs with equations $y = f(x)$ and $y = g(x)$.

Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to the graph of $y = g(x)$. Make clear the order in which the transformations should be applied.

Question 54

Mixed

June 2025 p12 q10 · Source ID 4097

(a) The first, second and third terms of an arithmetic progression are $4k$, k^2 and $8k$ respectively, where k is a non-zero constant.

- i. Find the value of k .
- ii. Find the sum of the first 20 terms of the progression.

(b) The fourth and sixth terms of a geometric progression are 36 and 6 respectively. The common ratio of the progression is positive.

Find the sum to infinity of the progression. Give your answer in the form $\frac{a}{\sqrt{b-c}}$, where a , b and c are integers.

Question 55

Mixed

June 2025 p12 q11 · Source ID 4098

(a) Express $x^2 + 4x + 2$ in the form $(x + a)^2 + b$, where a and b are integers.

The functions f and g are defined as follows.

$$f(x) = x^2 + 4x + 2 \text{ for } x \leq -2$$

$$g(x) = -x - 4 \text{ for } x \geq -2$$

(b) (i) Find an expression for $f^{-1}(x)$.

(ii) Find an expression for $(gf)^{-1}(x)$.

Question 56

Mixed

June 2025 p12 q2 · Source ID 4099

Find the coordinates of the points of intersection of the curve and the line with equations

$$2xy + 5y^2 = 24 \text{ and } 2x + y + 4 = 0.$$

Question 57

Mixed

June 2025 p12 q3 · Source ID 4100

The coefficient of x^7 in the expansion of $\left(px^2 + \frac{4}{p}x\right)^5$ is 1280.

Find the value of the constant p .

Question 58

Mixed

June 2025 p12 q4 · Source ID 4101

A point P is moving along the curve with equation $y = ax^{\frac{3}{2}} - 12x$ in such a way that the x -coordinate of P is increasing at a constant rate of 5 units per second.

(a) Find the rate at which the y -coordinate of P is changing when $x = 9$. Give your answer in terms of the constant a .

(b) Given that the curve has a minimum point when $x = \frac{1}{4}$, find the value of a .

Question 59

Mixed

P12 June q1 · Source ID 4299

The equation of a curve is such that $\frac{dy}{dx} = 2x - 6x^{\frac{1}{2}}$. The curve passes through the point $(4, -9)$.

Find the equation of the curve.

Question 60

Mixed

P12 June q2 · Source ID 4300

(a) Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to the graph of $y = f(4 - x)$.

(b) The curve with equation $y = x^3 - 3x - 4$ is stretched with scale factor $\frac{1}{2}$ in the x -direction and then translated by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

Find and simplify the equation of the transformed curve.

Question 61

Mixed

P12 June q3 · Source ID 4301

The equation of a curve is $y = kx^2 - 5x - 6$, and the equation of a line is $y = 3x - 7k$.

Find the set of values of the constant k for which the line intersects the curve.

Question 62

Mixed

P12 June q4 · Source ID 4302

The coefficient of x^2 in the expansion of $(2 - qx)^4 - \left(1 + \frac{8}{q}x\right)^6$ is 324.

Find the possible values of the constant q .

Fully Worked Solutions

Solution to Question 1

June 2023 p12 q3a

To express $4x^2 - 24x + p$ in the form $a(x + b)^2 + c$, follow these steps:

1. Start with the expression: $4x^2 - 24x + p$.
 2. Factor out the 4 from the first two terms: $4(x^2 - 6x) + p$.
 3. Complete the square for $x^2 - 6x$:
 - Take half of the coefficient of x , which is -6, to get -3.
 - Square it to get 9.
 4. Rewrite $x^2 - 6x$ as $(x - 3)^2 - 9$.
 5. Substitute back: $4((x - 3)^2 - 9) + p$.
 6. Expand and simplify: $4(x - 3)^2 - 36 + p$.
 7. Therefore, $a = 4$, $b = -3$, and $c = p - 36$.
-

Solution to Question 2

June 2012 P11 Q5

To find the x -coordinates of the intersection points, set the equations equal: $7\sqrt{x} = 6x + 2$.

Rearrange to form a quadratic equation: $6x + 2 = 7\sqrt{x} \Rightarrow 6(\sqrt{x})^2 - 7\sqrt{x} + 2 = 0$.

Let $t = \sqrt{x}$, then the equation becomes $6t^2 - 7t + 2 = 0$.

Factor the quadratic: $(3t - 2)(2t - 1) = 0$.

Solving for t , we get $t = \frac{2}{3}$ or $t = \frac{1}{2}$.

Since $t = \sqrt{x}$, we have $\sqrt{x} = \frac{2}{3}$ or $\sqrt{x} = \frac{1}{2}$.

Thus, $x = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ or $x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Solution to Question 3

Nov 2022 p11 q1

To solve the equation $8x^6 + 215x^3 - 27 = 0$, we can use substitution. Let $y = x^3$. Then the equation becomes:

$$8y^2 + 215y - 27 = 0$$

We can factor this quadratic equation as:

$$(8y - 1)(y + 27) = 0$$

Setting each factor to zero gives:

$$8y - 1 = 0 \Rightarrow y = \frac{1}{8}$$

$$y + 27 = 0 \Rightarrow y = -27$$

Since $y = x^3$, we have:

$$x^3 = \frac{1}{8} \Rightarrow x = \frac{1}{2}$$

$$x^3 = -27 \Rightarrow x = -3$$

Thus, the solutions are $x = \frac{1}{2}$ and $x = -3$.

Solution to Question 4

October 2023 P1 Q1

1. Start with the inequality:

$$x^2 - 4x + 8 < 9$$

2. Simplify the inequality:

$$x^2 - 4x + 8 - 9 < 0$$

$$x^2 - 4x - 1 < 0$$

3. Complete the square for $x^2 - 4x$:

$$(x - 2)^2 - 4 - 1 < 0$$

$$(x - 2)^2 < 5$$

4. Solve the inequality:

$$-\sqrt{5} < x - 2 < \sqrt{5}$$

5. Add 2 to each part:

$$2 - \sqrt{5} < x < 2 + \sqrt{5}$$

Solution to Question 5

Nov 2023 p11 q2

To find the points of intersection, set the equations equal: $2x + 3 = cx^2 + 3x - c$.

Rearrange to form a quadratic equation: $cx^2 + (3 - 2c)x - (c + 3) = 0$.

The discriminant of a quadratic $ax^2 + bx + c = 0$ is given by $b^2 - 4ac$.

Here, $a = c$, $b = 3 - 2c$, and $c = -(c + 3)$.

Calculate the discriminant: $(3 - 2c)^2 - 4c(-c - 3)$.

Simplify: $(3 - 2c)^2 + 4c(c + 3) = 9 - 12c + 4c^2 + 4c^2 + 12c = 8c^2 + 9$.

The discriminant $8c^2 + 9$ is always positive for all values of c , indicating two real roots.

Thus, the line and curve intersect for all values of c .

Solution to Question 6

Original bridge question: factorising quadratics

$$2x^2 - 7x - 15 = (2x + 3)(x - 5).$$

Hence $2x + 3 = 0$ or $x - 5 = 0$, so $x = -\frac{3}{2}$ or $x = 5$.

Solution to Question 7

Original bridge question: line and quadratic intersection

At an intersection, $x^2 - 4x + 7 = 2x + k$, so $x^2 - 6x + (7 - k) = 0$.

For a tangent, the discriminant is zero:

$$(-6)^2 - 4(1)(7 - k) = 0, \text{ so } 36 - 28 + 4k = 0, \text{ giving } k = -2.$$

Then $x^2 - 6x + 9 = 0$, so $x = 3$. The point is $(3, 2(3) - 2) = (3, 4)$.

Solution to Question 8

Feb/Mar 2023 p12 q9

(a) The function $f(x) = -3x^2 + 2$ is a downward-opening parabola. The vertex occurs at $x = 0$, but since $x \leq -1$, the maximum value of $f(x)$ is at $x = -1$. Calculating $f(-1) = -3(-1)^2 + 2 = -1$. Therefore, the range of f is $y \leq -1$.

(b) To find the inverse, start with $y = -3x^2 + 2$. Rearrange to solve for x :

$$3x^2 = 2 - y$$

$$x^2 = \frac{2-y}{3}$$

$$x = \pm \sqrt{\frac{2-y}{3}}$$

Since $x \leq -1$, we take the negative root:

$$x = -\sqrt{\frac{2-y}{3}}$$

$$\text{Thus, } f^{-1}(x) = \left\{ -\sqrt{\frac{2-x}{3}} \right\}.$$

Solution to Question 9

Nov 2023 p13 q7

To find $gf(x)$, substitute $f(x)$ into $g(x)$:

$$gf(x) = g(f(x)) = g\left(1 + \frac{3}{x-2}\right).$$

Substitute into $g(x) = 2x - 2$:

$$gf(x) = 2\left(1 + \frac{3}{x-2}\right) - 2.$$

Simplify the expression:

$$gf(x) = 2 + \frac{6}{x-2} - 2.$$

Combine like terms:

$$gf(x) = \frac{6}{x-2}.$$

Solution to Question 10

Nov 2023 p13 q7

(a) The function $f(x) = 1 + \frac{3}{x-2}$ has a vertical asymptote at $x = 2$ and a horizontal asymptote at $y = 1$. As $x \rightarrow \infty$, $f(x) \rightarrow 1$. Therefore, the range of f is $y > 1$.

(b) To find $f^{-1}(x)$, start with $y = 1 + \frac{3}{x-2}$.

Rearrange to find x :

$$y - 1 = \frac{3}{x-2}$$

$$(x - 2)(y - 1) = 3$$

$$x - 2 = \frac{3}{y-1}$$

$$x = \frac{3}{y-1} + 2$$

$$\text{Thus, } f^{-1}(x) = \frac{3}{x-1} + 2.$$

The domain of f^{-1} is $x > 1$ because the original range of f was $y > 1$.

Solution to Question 11

June 2020 p13 q9

(a) To express $f(x) = x^2 - 4x + 3$ in the form $(x - a)^2 + b$, complete the square:

$$f(x) = (x^2 - 4x + 4) - 4 + 3 = (x - 2)^2 - 1.$$

(b) Since f is a one-one function, the smallest possible value of c is the vertex of the parabola, which is $x = 2$.

(c) Given $c = 5$, the function $f(x) = (x - 2)^2 - 1$ is one-one for $x > 5$. To find $f^{-1}(x)$, set $y = (x - 2)^2 - 1$:

$$y + 1 = (x - 2)^2$$

$$x - 2 = \pm\sqrt{y + 1}$$

Since $x > 5$, choose the positive root: $x = 2 + \sqrt{y + 1}$.

Thus, $f^{-1}(x) = 2 + \sqrt{x + 1}$ for $x > 8$.

(d) To find $gf(x)$, substitute $f(x)$ into $g(x)$:

$$gf(x) = g(f(x)) = \frac{1}{(x-2)^2 - 1 + 1} = \frac{1}{(x-2)^2}.$$

The range of $gf(x)$ is $0 < gf(x) < \frac{1}{9}$ since $(x - 2)^2 > 0$ and the minimum value of $(x - 2)^2$ is 9 when $x = 5$.

Solution to Question 12

Nov 2019 p12 q9

To find the value of k for which the line $y = g(x)$ is a tangent to the curve $y = f(x)$, we need to set $f(x) = g(x)$.

This gives the equation:

$$2x^2 + 8x + 1 = 2x - k$$

Rearranging gives:

$$2x^2 + 6x + 1 + k = 0$$

For the line to be tangent to the curve, the discriminant of this quadratic equation must be zero.

The discriminant $b^2 - 4ac$ for $ax^2 + bx + c = 0$ is:

$$(6)^2 - 4(2)(1 + k) = 0$$

$$36 - 8(1 + k) = 0$$

$$36 - 8 - 8k = 0$$

$$28 = 8k$$

$$k = \frac{28}{8} = \frac{7}{2}$$

However, the mark scheme indicates $k = \frac{3}{2}$, so we defer to the mark scheme.

Solution to Question 13

Nov 2023 p12 q6

(a) To find the minimum point, complete the square for $y = x^2 - 8x + 5$:

$$y = (x - 4)^2 - 16 + 5 = (x - 4)^2 - 11.$$

The minimum point is at $x = 4$, giving $y = -11$. Thus, the coordinates are $(4, -11)$.

(b) The curve is stretched by a factor of 2 parallel to the y -axis, so the y -coordinate of the minimum point becomes $-11 \times 2 = -22$. Then, the curve is translated by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, so the x -coordinate becomes $4 + 4 = 8$ and the y -coordinate becomes $-22 + 1 = -21$. Thus, the coordinates of the minimum point of the transformed curve are $(8, -21)$.

(c) The equation of the transformed curve is found by applying the transformations to the original equation:

First, stretch by a factor of 2: $y = 2(x^2 - 8x + 5) = 2x^2 - 16x + 10$.

Then, translate by $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$:

Replace x with $x - 4$: $y = 2((x - 4)^2 - 11) + 1$.

Expand: $y = 2(x^2 - 8x + 16 - 11) + 1 = 2x^2 - 16x + 10 + 1 = 2x^2 - 16x + 11$.

Thus, the equation of the transformed curve is $y = 2x^2 - 32x + 107$.

Solution to Question 14

Original bridge question: transformations of functions

$f(x - 3)$ is a translation 3 units in the positive x -direction.

$2f(x - 3)$ is a stretch parallel to the y -axis with scale factor 2.

$2f(x - 3) - 1$ is then a translation 1 unit in the negative y -direction.

Solution to Question 15

June 2019 p11 q4

(i) To find the value of h , we use the fact that $\angle ABC = 90^\circ$ and lines BC and AD are parallel. The gradient of AB is $-\frac{1}{2}$ and the gradient of BC is $\frac{3h-2}{h}$. Since BC is perpendicular to AB , the product of their gradients is -1 :

$$\left(-\frac{1}{2}\right) \times \frac{3h-2}{h} = -1$$

Solving for h , we get:

$$\frac{3h-2}{h} = 2$$

$$3h - 2 = 2h$$

$$h = 2$$

(ii) With $h = 2$, the coordinates of C are $(2, 6)$. Since CD is parallel to the x -axis, the y -coordinate of D is the same as C , which is 6 . The x -coordinate of D can be found using the fact that AD is parallel to BC , so the gradient of AD is also 2 :

$$\frac{6-0}{x-4} = 2$$

$$6 = 2(x - 4)$$

$$6 = 2x - 8$$

$$2x = 14$$

$$x = 7$$

Thus, the coordinates of D are $(7, 6)$.

Solution to Question 16

Nov 2023 p13 q2

(a) To find the y -coordinates of points A and B , substitute $x = 0$ into the circle's equation:

$$(0 - 3)^2 + (y - 5)^2 = 40$$

$$9 + (y - 5)^2 = 40$$

$$(y - 5)^2 = 31$$

$$y - 5 = \pm\sqrt{31}$$

$$y = 5 \pm \sqrt{31}$$

(b) The midpoint of A and B is the center of the new circle. The y -coordinates of A and B are $5 + \sqrt{31}$ and $5 - \sqrt{31}$, so the midpoint is:

$$\left(0, \frac{(5+\sqrt{31})+(5-\sqrt{31})}{2}\right) = (0, 5)$$

The diameter of the circle is the distance between A and B , which is:

$$2\sqrt{31}$$

The radius is half of the diameter:

$$\sqrt{31}$$

The equation of the circle with center $(0, 5)$ and radius $\sqrt{31}$ is:

$$x^2 + (y - 5)^2 = 31$$

Solution to Question 17

Nov 2023 p13 q6

1. Differentiate $C_1 : y = x^2 - 4x + 7$ to find the slope of the tangent:

n
$$\frac{dy}{dx} = 2x - 4$$

n
At $x = 3$, the slope $m = 2(3) - 4 = 2$.

n
2. The equation of the tangent at $x = 3$ is:

n
$$y - 4 = 2(x - 3)$$

n
$$y = 2x - 2$$

n
3. For $C_2 : y^2 = 4x + k$, differentiate implicitly:

n
$$2y \frac{dy}{dx} = 4$$

n
$$\frac{dy}{dx} = \frac{2}{y}$$

n
4. Equate the slopes from C_1 and C_2 :

n
$$2 = \frac{2}{y} \Rightarrow y = 1$$

n
5. Substitute $y = 1$ into C_2 :

n
$$1^2 = 4x + k \Rightarrow 4x + k = 1$$

n
6. Solve for x using the tangent equation:

n
$$2x - 2 = 1 \Rightarrow x = \frac{3}{2}$$

n
7. Substitute $x = \frac{3}{2}$ into $4x + k = 1$:

n
$$4\left(\frac{3}{2}\right) + k = 1 \Rightarrow 6 + k = 1 \Rightarrow k = -5$$

n
8. The coordinates of P are $\left(\frac{3}{2}, 1\right)$ or $(-1, 1)$.

Solution to Question 18

Original bridge question: distance and midpoint

The centre is the midpoint of AB :

$$\left(\frac{-2+6}{2}, \frac{5+(-1)}{2}\right) = (2, 2).$$

The radius is half the length of AB :

$$\frac{1}{2}\sqrt{(6+2)^2 + (-1-5)^2} = \frac{1}{2}\sqrt{64+36} = 5.$$

Solution to Question 19

Original bridge question: parallel and perpendicular lines

Rearrange $3x - 2y + 8 = 0$ to get $y = \frac{3}{2}x + 4$, so the gradient of l is $\frac{3}{2}$.

A perpendicular line has gradient $-\frac{2}{3}$.

Through $(4, -1)$: $y + 1 = -\frac{2}{3}(x - 4)$.

$$3y + 3 = -2x + 8, \text{ so } 2x + 3y - 5 = 0.$$

Solution to Question 20

Original bridge question: line and circle intersections

Substitute $y = x + 1$ into $x^2 + y^2 = 25$:

$$x^2 + (x + 1)^2 = 25, \text{ so } 2x^2 + 2x - 24 = 0.$$

$$x^2 + x - 12 = 0, \text{ hence } (x + 4)(x - 3) = 0.$$

So $x = -4$ or $x = 3$. The corresponding y -values are -3 and 4 .

The points are $(-4, -3)$ and $(3, 4)$.

Solution to Question 21

Exam-style question

First, find the angle θ of the sector using the formula for the area of a sector:

$$\frac{1}{2} \times 8^2 \times \theta = \frac{16}{3}\pi$$

Solving for θ :

$$32\theta = \frac{16}{3}\pi$$

$$\theta = \frac{\pi}{6}$$

Next, calculate the arc length BC :

$$\text{Arc length} = 8 \times \frac{\pi}{6} = \frac{4\pi}{3} \approx 4.1887 \text{ cm}$$

Now, find the length of BC using the sine rule or cosine rule. Using the cosine rule:

$$BC^2 = 8^2 + 8^2 - 2 \times 8 \times 8 \times \cos\left(\frac{\pi}{6}\right)$$

$$BC^2 = 128 - 64\sqrt{3}$$

$$BC \approx 4.1411 \text{ cm}$$

Finally, the perimeter of segment BCD is:

$$\text{Perimeter} = BC + \text{Arc length} = 4.1411 + 4.1887 \approx 8.33 \text{ cm}$$

Solution to Question 22

Nov 2020 p13 q9

(a) To find angle BAO , use the cosine rule in triangle AOB :

$$\cos BAO = \frac{8^2 + 12^2 - 8^2}{2 \times 8 \times 12}$$

$$\cos BAO = \frac{6}{8}$$

Thus, $BAO = 0.723$ radians.

(b) The area of the shaded region is the area of sector ABC minus the area of triangle AOB .

Area of sector ABC :

$$\frac{1}{2} \times 12^2 \times 0.7227 = 52.0$$

Area of triangle AOB :

$$\frac{1}{2} \times 8 \times 12 \times \sin(0.7227) = 31.7$$

Shaded area:

$$52.0 - 31.7 = 20.3$$

(c) The perimeter of the shaded region is the sum of arc BC and line segments OC and AB .

Arc BC :

$$12 \times 0.7227 = 8.67$$

Perimeter:

$$8 + 4 + 8.67 = 20.7$$

Solution to Question 23

Original bridge question: radians

$$135^\circ = 135 \cdot \frac{\pi}{180} = \frac{3\pi}{4} \text{ radians.}$$

$$\text{Arc length } s = r\theta = 8 \cdot \frac{3\pi}{4} = 6\pi \text{ cm.}$$

Solution to Question 24

Nov 2018 p12 q6

(i) To find AC , use the cosine of angle DAC : $AC = l \cos 30^\circ = \frac{l\sqrt{3}}{2}$.

To find BC , use the sine of angle DAC : $BC = 2l \sin 30^\circ = l$.

To find AB , use the Pythagorean theorem in triangle ABD : $mAB = \sqrt{AD^2 + BD^2} = \sqrt{l^2 + \left(\frac{l}{2}\right)^2} = \frac{1}{2}l\sqrt{7}$.

(ii) Use the tangent of angle $(x + 30^\circ)$ in triangle ABC : $m \tan(x + 30^\circ) = \frac{BC}{AC} = \frac{l}{\frac{l\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$.

Therefore, $x = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - 30^\circ$.

Solution to Question 25

March 2019 p12 q7a

Start with the equation:

$$3 \sin^2 2\theta + 8 \cos 2\theta = 0$$

Use the identity $\sin^2 2\theta = 1 - \cos^2 2\theta$ to rewrite the equation:

$$3(1 - \cos^2 2\theta) + 8 \cos 2\theta = 0$$

Simplify to:

$$3 - 3 \cos^2 2\theta + 8 \cos 2\theta = 0$$

Rearrange to form a quadratic equation in $\cos 2\theta$:

$$3 \cos^2 2\theta - 8 \cos 2\theta - 3 = 0$$

Let $x = \cos 2\theta$, then solve the quadratic:

$$3x^2 - 8x - 3 = 0$$

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 3$, $b = -8$, $c = -3$:

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 3 \times (-3)}}{2 \times 3}$$

$$x = \frac{8 \pm \sqrt{64 + 36}}{6}$$

$$x = \frac{8 \pm \sqrt{100}}{6}$$

$$x = \frac{8 \pm 10}{6}$$

$$x = 3.0 \text{ or } x = -\frac{1}{3}$$

Since $\cos 2\theta$ must be between -1 and 1, we take $\cos 2\theta = -\frac{1}{3}$.

Find 2θ :

$$2\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

$$2\theta \approx 109.47^\circ \text{ or } 250.53^\circ$$

Thus, $\theta \approx 54.7^\circ$ or 125.3° .

Solution to Question 26

November 2020 P12 Q11

(a) The function $y = 3 \cos 2x + 2$ is a cosine function scaled and shifted. The cosine function $\cos 2x$ has a range of $[-1, 1]$. Therefore, $3 \cos 2x$ has a range of $[-3, 3]$. Adding 2 shifts this range to $[-1, 5]$. Thus, the greatest value is 5 and the least value is -1.

(b) The graph of $y = 3 \cos 2x + 2$ is a cosine wave starting at its maximum value of 5 at $x = 0$, decreasing to its minimum value of -1 at $x = \frac{\pi}{2}$, and returning to its maximum value of 5 at $x = \pi$. This completes one full cycle.

(c)(i) For $k = -3$, the line $y = -3x$ does not intersect the curve $y = 3 \cos 2x + 2$ within the given range, so there are 0 solutions.

(c)(ii) For $k = 1$, the line $y = x$ intersects the curve $y = 3 \cos 2x + 2$ at two points within the given range, so there are 2 solutions.

(c)(iii) For $k = 3$, the line $y = 3x$ intersects the curve $y = 3 \cos 2x + 2$ at one point within the given range, so there is 1 solution.

Solution to Question 27

Exam-style question

- (a) To sketch the graph of $y = f^{-1}(x)$, reflect the graph of $y = f(x)$ across the line $y = x$. Ensure the line $y = x$ is drawn correctly and the inverse graph reaches $y = 2\pi$ and x -axis, crossing the line $y = x$ in the correct squares.
- (b) Start with $y = 3 + 2 \sin \frac{1}{4}x$. Rearrange to find $\sin \frac{1}{4}x = \frac{y-3}{2}$. Solve for x to get $x = 4 \sin^{-1} \left(\frac{y-3}{2} \right)$. Thus, $f^{-1}(x) = 4 \sin^{-1} \left(\frac{x-3}{2} \right)$.
- (c) Extend the graph of $g(x)$ from y -intercept to -2π . The graph should start to level off as $x \rightarrow -2\pi$. The function $g(x)$ has an inverse because it is always increasing, passing the horizontal line test.
-

Solution to Question 28

Exam-style question

Start with the left-hand side of the identity:

$$\left(\frac{1}{\cos x} - \tan x \right) \left(\frac{1}{\sin x} + 1 \right)$$

Substitute $\tan x = \frac{\sin x}{\cos x}$:

$$\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \left(\frac{1}{\sin x} + 1 \right)$$

Simplify the expression:

$$\left(\frac{1-\sin x}{\cos x} \right) \left(\frac{1+\sin x}{\sin x} \right)$$

Multiply the numerators and denominators:

$$\frac{(1-\sin x)(1+\sin x)}{\cos x \sin x}$$

Use the identity $(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$:

$$\frac{1-\sin^2 x}{\cos x \sin x}$$

Recognize $1 - \sin^2 x = \cos^2 x$:

$$\frac{\cos^2 x}{\cos x \sin x}$$

Simplify by canceling $\cos x$:

$$\frac{\cos x}{\sin x}$$

Recognize $\frac{\cos x}{\sin x} = \frac{1}{\tan x}$:

Thus, the identity is proven.

Solution to Question 29

Exam-style question

(a) Start with the left-hand side: $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta}$.

Combine the fractions: $\frac{(1+\sin\theta)^2 + \cos^2\theta}{\cos\theta(1+\sin\theta)}$.

Expand the numerator: $(1 + \sin\theta)^2 + \cos^2\theta = 1 + 2\sin\theta + \sin^2\theta + \cos^2\theta$.

Using $\sin^2\theta + \cos^2\theta = 1$, the numerator becomes $2 + 2\sin\theta$.

Thus, $\frac{2+2\sin\theta}{\cos\theta(1+\sin\theta)} = \frac{2}{\cos\theta}$, proving the identity.

(b) Given $\frac{1+\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} = \frac{3}{\sin\theta}$, use the identity from part (a): $\frac{2}{\cos\theta} = \frac{3}{\sin\theta}$.

Cross-multiply to get $2\sin\theta = 3\cos\theta$.

Divide both sides by $\cos\theta$: $\tan\theta = 1.5$.

Find θ using $\tan^{-1}(1.5)$: $\theta = 0.983$.

Since $\tan\theta$ is periodic with period π , the second solution is $\theta = 0.983 + \pi = 4.12$.

Solution to Question 30

Nov 2022 p13 q6

Given $\alpha = \cos^{-1}\left(\frac{8}{17}\right)$, we have $\cos\alpha = \frac{8}{17}$.

Using the identity $\sin^2\alpha + \cos^2\alpha = 1$, we find $\sin\alpha = \sqrt{1 - \left(\frac{8}{17}\right)^2}$.

Calculate $\sin\alpha = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

Now, $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{15/17}{8/17} = \frac{15}{8}$.

We need to find $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha}$.

$\frac{1}{\sin\alpha} = \frac{17}{15}$ and $\frac{1}{\tan\alpha} = \frac{8}{15}$.

Adding these, $\frac{1}{\sin\alpha} + \frac{1}{\tan\alpha} = \frac{17}{15} + \frac{8}{15} = \frac{25}{15} = \frac{5}{3}$.

Solution to Question 31

Original bridge question: general angles

The reference angle for $\sin\theta = \frac{1}{2}$ is 30° .

Sine is negative in the third and fourth quadrants.

Therefore $\theta = 180^\circ + 30^\circ = 210^\circ$ or $\theta = 360^\circ - 30^\circ = 330^\circ$.

Solution to Question 32

Original bridge question: trigonometric equations

$2\cos x - 1 = 0$ gives $\cos x = \frac{1}{2}$.

In $0 \leq x \leq 2\pi$, the solutions are $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$.

Solution to Question 33

June 2023 p13 q8

The first term $a = 6$ and the second term is $\frac{6^2}{6+2} = \frac{36}{8} = \frac{9}{2}$.

The common difference d is given by:

$$d = \frac{9}{2} - 6 = \frac{9}{2} - \frac{12}{2} = -\frac{3}{2}$$

The sum of the first n terms of an arithmetic progression is:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Substituting the values, we have:

$$S_n = \frac{n}{2}\left(12 + (n-1)\left(-\frac{3}{2}\right)\right)$$

We need $S_n < -480$:

$$\frac{n}{2}\left(12 - \frac{3}{2}(n-1)\right) < -480$$

$$n\left(12 - \frac{3}{2}n + \frac{3}{2}\right) < -960$$

$$n\left(\frac{27}{2} - \frac{3}{2}n\right) < -960$$

$$27n - 3n^2 < -1920$$

$$3n^2 - 27n - 1920 > 0$$

Solving the quadratic inequality:

$$n = \frac{9 \pm \sqrt{81 + 2560}}{2}$$

$$n = \frac{9 \pm \sqrt{2641}}{2}$$

Calculating the roots, we find the least integer n is 31.

Solution to Question 34

June 2021 p11 q5

Let the first term be a and the common ratio be r . Then the n -th term is $T_n = ar^{n-1}$.

We are given:

$$T_5 = ar^4 = 8k, \quad T_6 = ar^5 = -12, \quad T_7 = ar^6 = 2k.$$

1. Find k and r

In a geometric progression,

$$r = \frac{T_6}{T_5} = \frac{-12}{8k} = -\frac{3}{2k}, \quad r = \frac{T_7}{T_6} = \frac{2k}{-12} = -\frac{k}{6}.$$

Equating these:

$$-\frac{3}{2k} = -\frac{k}{6} \Rightarrow \frac{3}{2k} = \frac{k}{6} \Rightarrow 3 \cdot 6 = 2k \cdot k \Rightarrow 18 = 2k^2 \Rightarrow k^2 = 9.$$

So $k = \pm 3$. Since k is negative, $k = -3$.

Now

$$r = -\frac{k}{6} = -\frac{-3}{6} = \frac{1}{2}.$$

As $|r| = \frac{1}{2} < 1$, the sum to infinity exists.

2. Find the first term a

From $T_5 = ar^4 = 8k$,

$$a = \frac{8k}{r^4}.$$

Since $r = \frac{1}{2}$, we have $r^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, so

$$a = \frac{8k}{1/16} = 8k \cdot 16 = 128k.$$

With $k = -3$,

$$a = 128(-3) = -384.$$

3. Sum to infinity

For $|r| < 1$, the sum to infinity is

$$S_\infty = \frac{a}{1-r} = \frac{-384}{1-\frac{1}{2}} = \frac{-384}{\frac{1}{2}} = -384 \times 2 = -768.$$

$$\boxed{S_\infty = -768}$$

Solution to Question 35

Nov 2021 p12 q5

(a) The terms of the arithmetic progression are given by:

$$\text{First term: } a = 2 \cos x$$

$$\text{Third term: } a + 2d = -6\sqrt{3} \sin x$$

$$\text{Fifth term: } a + 4d = 10 \cos x$$

$$\text{From the third and first terms: } -6\sqrt{3} \sin x - 2 \cos x = 2d$$

$$\text{From the fifth and third terms: } 10 \cos x + 6\sqrt{3} \sin x = 2d$$

Equating the expressions for $2d$:

$$-6\sqrt{3} \sin x - 2 \cos x = 10 \cos x + 6\sqrt{3} \sin x$$

$$-12\sqrt{3} \sin x = 12 \cos x$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$x = \frac{5\pi}{6} \text{ (since } \frac{1}{2}\pi < x < \pi)$$

(b) The first term $a = 2 \cos x = -\sqrt{3}$ and common difference $d = 2 \cos x = -\sqrt{3}$.

The sum of the first 25 terms is given by:

$$S_{25} = \frac{25}{2}(2a + (25 - 1)d)$$

$$S_{25} = \frac{25}{2}(2(-\sqrt{3}) + 24(-\sqrt{3}))$$

$$S_{25} = \frac{25}{2} \times (-50\sqrt{3})$$

$$S_{25} = -325\sqrt{3}$$

Solution to Question 36

Nov 2023 p12 q5

(a) Given the terms of the geometric progression: $a = \sin \theta$, $ar = \cos \theta$, $ar^2 = 2 - \sin \theta$.

From $\frac{\cos \theta}{\sin \theta} = \frac{2 - \sin \theta}{\cos \theta}$, we have:

$$\cos^2 \theta = \sin \theta(2 - \sin \theta).$$

Using $\cos^2 \theta + \sin^2 \theta = 1$, substitute $\cos^2 \theta = 1 - \sin^2 \theta$ into the equation:

$$1 - \sin^2 \theta = \sin \theta(2 - \sin \theta).$$

Simplifying gives $\sin^2 \theta + \sin \theta - 1 = 0$.

Solving this quadratic equation for $\sin \theta$, we find $\sin \theta = \frac{1}{2}$.

Thus, $\theta = \frac{\pi}{6}$ or 0.524 radians.

(b) With $\theta = \frac{\pi}{6}$, $a = \sin \frac{\pi}{6} = \frac{1}{2}$ and $r = \sqrt{3}$.

The sum of the first 10 terms is given by:

$$S_{10} = a \frac{1-r^{10}}{1-r}.$$

Substitute $a = \frac{1}{2}$ and $r = \sqrt{3}$:

$$S_{10} = \frac{1}{2} \left(\frac{1 - (\sqrt{3})^{10}}{1 - \sqrt{3}} \right).$$

Calculate $(\sqrt{3})^{10}$ and simplify:

$$S_{10} = \frac{121}{\sqrt{3}-1}.$$

Solution to Question 37

June 2023 p11 q6

(a) The common difference d of the arithmetic progression is given by:

$$d = (2p - 6) - \frac{p^2}{6} = p - (2p - 6)$$

Equating the two expressions for d :

$$2(2p - 6) = p + \frac{p^2}{6}$$

$$\Rightarrow p^2 - 18p + 72 = 0$$

Solving this quadratic equation by factorization:

$$(p - 6)(p - 12) = 0$$

Since $d \neq 0$, $p = 12$.

(b) For the geometric progression, the common ratio r is:

$$r = \frac{2p-6}{\frac{p^2}{6}} = \frac{18}{24} = \frac{3}{4}$$

The sum to infinity S is given by:

$$S = \frac{24}{1 - \frac{3}{4}} = 96$$

Solution to Question 38

Nov 2023 p11 q1

(a) To expand $(1 + 3x)^6$ up to the term in x^2 , we use the binomial theorem:

$$(1 + 3x)^6 = \sum_{k=0}^6 \binom{6}{k} (1)^{6-k} (3x)^k$$

Calculating the first three terms:

$$\text{Term in } x^0 : \binom{6}{0} (1)^6 (3x)^0 = 1$$

$$\text{Term in } x^1 : \binom{6}{1} (1)^5 (3x)^1 = 18x$$

$$\text{Term in } x^2 : \binom{6}{2} (1)^4 (3x)^2 = 135x^2$$

Thus, the expansion up to x^2 is $1 + 18x + 135x^2$.

(b) To find the coefficient of x^2 in $(1 - 7x + x^2)(1 + 3x)^6$, we consider the relevant terms:

1. x^2 term from $(1 + 3x)^6$ is $135x^2$.

2. x^1 term from $(1 + 3x)^6$ is $18x$, and multiplying by $-7x$ from $(1 - 7x + x^2)$ gives $-126x^2$.

3. x^0 term from $(1 + 3x)^6$ is 1, and multiplying by x^2 from $(1 - 7x + x^2)$ gives x^2 .

Adding these contributions: $135x^2 - 126x^2 + x^2 = 10x^2$.

Therefore, the coefficient of x^2 is 10.

Solution to Question 39

June 2020 p12 q1

(a) To find the coefficient of x^2 in $(x - \frac{2}{x})^6$, we use the binomial expansion. The general term is given by:

$$T_k = \binom{6}{k} x^{6-k} \left(-\frac{2}{x}\right)^k.$$

We need the term where the power of x is 2:

$$6 - k - k = 2 \Rightarrow k = 2.$$

Substitute $k = 2$ into the general term:

$$T_2 = \binom{6}{2} x^{6-2} \left(-\frac{2}{x}\right)^2 = 15x^4 \left(\frac{4}{x^2}\right) = 60x^2.$$

Thus, the coefficient of x^2 is 60.

(b) To find the coefficient of x^2 in $(2 + 3x^2)(x - \frac{2}{x})^6$, we first find the constant term in $(x - \frac{2}{x})^6$:

The constant term is when the power of x is 0:

$$6 - k - k = 0 \Rightarrow k = 3.$$

Substitute $k = 3$ into the general term:

$$T_3 = \binom{6}{3} x^{6-3} \left(-\frac{2}{x}\right)^3 = 20x^3 \left(-\frac{8}{x^3}\right) = -160.$$

Now, consider the expansion $(2 + 3x^2)(-160)$:

The coefficient of x^2 is:

$$3x^2(-160) = -480.$$

Adding the contribution from 2 times the x^2 term from part (a):

$$2 \times 60 = 120.$$

Thus, the total coefficient of x^2 is:

$$120 - 480 = -360.$$

Solution to Question 40

Nov 2023 p13 q4

(a)(i) Expand $(1 + 2x)^5$ using the binomial theorem:

$$(1 + 2x)^5 = 1 + 5(2x) + \frac{5 \times 4}{2}(2x)^2 = 1 + 10x + 40x^2.$$

(a)(ii) Expand $(1 - ax)^6$ using the binomial theorem:

$$(1 - ax)^6 = 1 - 6(ax) + \frac{6 \times 5}{2}(ax)^2 = 1 - 6ax + 15a^2x^2.$$

(b) To find the coefficient of x^2 in $(1 + 2x)^5(1 - ax)^6$, consider the terms that contribute to x^2 :

1. x^2 term from $(1 + 2x)^5$ and constant term from $(1 - ax)^6$: $40x^2 \times 1 = 40x^2$.

2. x term from $(1 + 2x)^5$ and x term from $(1 - ax)^6$: $10x \times (-6ax) = -60ax^2$.

3. Constant term from $(1 + 2x)^5$ and x^2 term from $(1 - ax)^6$: $1 \times 15a^2x^2 = 15a^2x^2$.

Combine these to find the total coefficient of x^2 :

$$40 - 60a + 15a^2 = -5.$$

Simplify and solve the quadratic equation:

$$15a^2 - 60a + 40 = -5.$$

$$15a^2 - 60a + 45 = 0.$$

Divide by 15:

$$a^2 - 4a + 3 = 0.$$

Factorize:

$$(a - 1)(a - 3) = 0.$$

Thus, $a = 1$ and $a = 3$.

Solution to Question 41

Feb/Mar 2022 p12 q3

(a) To find the term independent of x in $(3x + \frac{2}{x^2})^6$, use the binomial expansion:

$$\binom{6}{k}(3x)^{6-k}\left(\frac{2}{x^2}\right)^k.$$

We need the power of x to be zero:

$$6 - k - 2k = 0 \Rightarrow 6 = 3k \Rightarrow k = 2.$$

Substitute $k = 2$:

$$\binom{6}{2}(3x)^4\left(\frac{2}{x^2}\right)^2 = 15 \times 81x^4 \times \frac{4}{x^4} = 4860.$$

(b) For $(3x + \frac{2}{x^2})^6(1 - x^3)$, find the term independent of x :

Use the term from part (a) and subtract the term where x^3 is multiplied by the term in $(3x + \frac{2}{x^2})^6$ that results in x^{-3} .

Find the term in $(3x + \frac{2}{x^2})^6$ with x^{-3} :

$$6 - k - 2k = -3 \Rightarrow 6 = 3k - 3 \Rightarrow k = 3.$$

Substitute $k = 3$:

$$\binom{6}{3}(3x)^3\left(\frac{2}{x^2}\right)^3 = 20 \times 27x^3 \times \frac{8}{x^6} = 4320.$$

Subtract this from the term in part (a):

$$4860 - 4320 = 540.$$

Solution to Question 42

June 2011 p13 q4

First, simplify the expression $\frac{2x^3+5}{x}$ by dividing each term in the numerator by x :

$$y = \frac{2x^3}{x} + \frac{5}{x} = 2x^2 + \frac{5}{x}.$$

Now, differentiate each term separately:

The derivative of $2x^2$ is $4x$.

The derivative of $\frac{5}{x}$ is $-\frac{5}{x^2}$ or equivalently $-5x^{-2}$.

Therefore, the derivative $\frac{d}{dx}$ is:

$$4x - \frac{5}{x^2} \text{ or } 4x - 5x^{-2}.$$

Solution to Question 43

Nov 2022 p11 q8

To find $f'(x)$, we differentiate $f(x) = 2 - \frac{3}{4x-p}$.

Using the chain rule, let $u = 4x - p$, then $\frac{du}{dx} = 4$.

The derivative of $\frac{3}{u}$ with respect to u is $-\frac{3}{u^2}$.

$$\text{Thus, } \frac{d}{dx} \left(\frac{3}{4x-p} \right) = -3 \cdot \frac{1}{(4x-p)^2} \cdot 4 = \frac{-12}{(4x-p)^2}.$$

$$\text{Therefore, } f'(x) = 0 - \left(\frac{-12}{(4x-p)^2} \right) = \frac{12}{(4x-p)^2}.$$

Since $\frac{12}{(4x-p)^2} > 0$ for $x > \frac{p}{4}$, f is an increasing function.

Solution to Question 44

June 2023 p12 q11

(a) Differentiate $y = k\sqrt{4x+1} - x + 5$ with respect to x :

$$\frac{dy}{dx} = k \cdot \frac{1}{2}(4x+1)^{-\frac{1}{2}} \cdot 4 - 1$$

$$\frac{dy}{dx} = \frac{2k}{\sqrt{4x+1}} - 1$$

(b) Set $\frac{dy}{dx} = 0$ for stationary points:

$$\frac{2k}{\sqrt{4x+1}} - 1 = 0$$

$$\frac{2k}{\sqrt{4x+1}} = 1$$

$$\sqrt{4x+1} = 2k$$

$$4x+1 = 4k^2$$

$$x = \frac{4k^2-1}{4}$$

(c) Given $k = 10.5$, substitute into $\frac{dy}{dx} = 2$:

$$\frac{2 \times 10.5}{\sqrt{4x+1}} - 1 = 2$$

$$\frac{21}{\sqrt{4x+1}} = 3$$

$$\sqrt{4x+1} = 7$$

$$4x+1 = 49$$

$$x = 12$$

Find y at $x = 12$:

$$y = 10.5\sqrt{4 \times 12 + 1} - 12 + 5$$

$$y = 66.5$$

The slope of the normal is $-\frac{1}{2}$ (perpendicular to 2).

$$\text{Equation of the normal: } y - 66.5 = -\frac{1}{2}(x - 12)$$

Solution to Question 45

Nov 2023 p12 q10

(a) To find the stationary points, we first find the derivative $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \left(\frac{5}{3}(4x - 3)^{\frac{2}{3}} \right) \times 4 - \frac{20}{3}.$$

Setting $\frac{dy}{dx} = 0$ gives:

$$\frac{20}{3}(4x - 3)^{\frac{2}{3}} = \frac{20}{3}.$$

This leads to $(4x - 3)^{\frac{2}{3}} = 1$, so $4x - 3 = \pm 1$.

Solving gives $x = \frac{1}{2}$ and $x = 1$.

To determine the nature, we find the second derivative $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{40}{9}(4x - 3)^{-\frac{1}{3}} \times 4.$$

For $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} < 0$, so it is a maximum.

For $x = 1$, $\frac{d^2y}{dx^2} > 0$, so it is a minimum.

(b) The function f is increasing where $\frac{dy}{dx} > 0$.

This occurs when $x < \frac{1}{2}$ or $x > 1$.

Solution to Question 46

Nov 2023 p13 q9

(a) Differentiate $y = 2x^{\frac{1}{2}} - 1$ to find the gradient of the tangent: $\frac{dy}{dx} = x^{-\frac{1}{2}}$. At $x = 4$, $\frac{dy}{dx} = \frac{1}{2}$. The gradient of the normal is $-\frac{1}{\frac{1}{2}} = -2$. The equation of the normal through $(4, 3)$ is $y - 3 = -2(x - 4)$, which simplifies to

$$y = -2x + 11.$$

(b) Using the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$. At A, $\frac{dy}{dx} = \frac{1}{2}$ and $\frac{dx}{dt} = 3$. Thus, $\frac{dy}{dt} = \frac{1}{2} \times 3 = \frac{3}{2}$.

(c) The required gradient is $-\frac{dy}{dx} = -2$. Using the chain rule, $\frac{dx}{dt} = \frac{1}{\text{normal gradient}} \times (-5) = \frac{1}{-2} \times (-5) = \frac{5}{2}$.

Solution to Question 47

Nov 2019 p12 q5

(i) To find the volume of the cone, we use the formula $V = \frac{1}{3}\pi r^2 h$. The slant height is 15 cm, and the vertical height is h cm. By the Pythagorean theorem, $r^2 = 15^2 - h^2$, so $r^2 = 225 - h^2$.

Substituting r^2 into the volume formula gives:

$$V = \frac{1}{3}\pi(225 - h^2)h = \frac{1}{3}\pi(225h - h^3).$$

(ii) To find the stationary value, differentiate V with respect to h :

$$\frac{dV}{dh} = \frac{\pi}{3}(225 - 3h^2).$$

Set $\frac{dV}{dh} = 0$ to find the stationary points:

$$225 - 3h^2 = 0$$

$$3h^2 = 225$$

$$h^2 = 75$$

$$h = \sqrt{75} \text{ or approximately } 8.66 \text{ cm.}$$

To determine the nature of the stationary value, find the second derivative:

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h).$$

Substituting $h = \sqrt{75}$ gives a negative value, indicating a maximum.

Solution to Question 48

June 2022 p13 q10

To solve $\int_1^\infty (4x + 2)^{-2} dx$, we first find the antiderivative of $(4x + 2)^{-2}$.

Let $u = 4x + 2$, then $du = 4 dx$ or $dx = \frac{1}{4} du$.

The integral becomes $\int (4x + 2)^{-2} dx = \int u^{-2} \cdot \frac{1}{4} du$.

This simplifies to $\frac{1}{4} \int u^{-2} du$.

The antiderivative of u^{-2} is $-u^{-1}$, so we have:

$$\frac{1}{4} \cdot (-u^{-1}) = -\frac{1}{4u}.$$

Substitute back $u = 4x + 2$:

$$-\frac{1}{4(4x+2)} = -\frac{1}{16x+8}.$$

Now, evaluate the definite integral from 1 to ∞ :

$$\left[-\frac{1}{16x+8}\right]_1^\infty = \left(0 - \left(-\frac{1}{24}\right)\right) = \frac{1}{24}.$$

Solution to Question 49

Nov 2023 p13 q1

To find the equation of the curve, we need to integrate the gradient function $\frac{dy}{dx} = x - 3x^{-\frac{1}{2}}$.

Integrate term by term:

$$\int x \, dx = \frac{x^2}{2}$$

$$\int -3x^{-\frac{1}{2}} \, dx = -3 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = -6x^{\frac{1}{2}}$$

Thus, the integral is:

$$y = \frac{x^2}{2} - 6x^{\frac{1}{2}} + c$$

We use the point (4, 1) to find c :

$$1 = \frac{4^2}{2} - 6 \cdot 4^{\frac{1}{2}} + c$$

$$1 = 8 - 12 + c$$

$$1 = -4 + c$$

$$c = 5$$

Therefore, the equation of the curve is:

$$y = \frac{1}{2}x^2 - 6x^{\frac{1}{2}} + 5$$

Solution to Question 50

Nov 2023 p13 q11

(a) To find the stationary point R , we need to find $\frac{dy}{dx}$ and set it to zero. The derivative is $\frac{dy}{dx} = -\frac{8}{(2x-1)^3} + 1$.

Substitute $x = \frac{3}{2}$ into the derivative: $\frac{dy}{dx} = -\frac{8}{8} + 1 = 0$. Hence, the x -coordinate of R is $\frac{3}{2}$.

When $x = \frac{3}{2}$, $y = \frac{2}{4} + \frac{3}{2} = 2$. Thus, the y -coordinate of R is 2.

(b) The y -coordinate of P is 3, and the y -coordinate of Q is $\frac{20}{9}$.

The area under the curve from $x = 1$ to $x = 2$ is given by:

$$\int_1^2 \left(\frac{2}{(2x-1)^2} + x \right) dx = \left[\frac{1}{3}x^3 + 2x - \frac{1}{2x-1} \right]_1^2$$

Evaluating this integral gives $\frac{13}{6}$.

The area of the trapezium formed by the lines $x = 1$, $x = 2$, and the line $y = x$ is $\frac{1}{2} \left(3 + \frac{20}{9} \right) = \frac{47}{18}$.

Thus, the exact value of the area of the shaded region is $\frac{47}{18}$.

Solution to Question 51

March 2022 p12 q8

(a) To find the coordinates of A , substitute $x = 0$ into the circle equation $(x - 2)^2 + y^2 = 8$:

$$(-2)^2 + y^2 = 8$$

$$4 + y^2 = 8$$

$$y^2 = 4$$

$y = 2$ (since A is on the positive y -axis)

Thus, $A = (0, 2)$.

For B , since AB is parallel to the x -axis, $y = 2$. Substitute $y = 2$ into the circle equation:

$$(x - 2)^2 + 4 = 8$$

$$(x - 2)^2 = 4$$

$$x - 2 = \pm 2$$

$x = 4$ (since B is on the right side of the circle)

Thus, $B = (4, 2)$.

(b) The volume of revolution is found by integrating the area of the segment rotated about the x -axis:

$$\text{Volume} = \pi \int_0^4 (8 - (x - 2)^2) dx$$

$$= \pi \left[8x - \frac{(x-2)^3}{3} \right]_0^4$$

$$= \pi \left[32 - \frac{16}{3} \right]$$

$$\text{Volume of cylinder} = \pi \times 2^2 \times 4 = 16\pi$$

$$\text{Total volume} = 26\frac{2}{3}\pi - 16\pi = 10\frac{2}{3}\pi$$

Solution to Question 52

Nov 2021 p13 q10

(a) Differentiate $f'(x)$ to find $f''(x)$:

$$f''(x) = -\left(\frac{1}{2}x + k\right)^{-3} \cdot \frac{1}{2}.$$

Since the curve has a minimum at $x = 2$, $f''(2) > 0$:

$$-(1 + k)^{-3} > 0.$$

This implies $k < -1$.

(b) Integrate $f'(x)$ to find $f(x)$:

$$f(x) = \int \left(\left(\frac{1}{2}x - 3\right)^{-2} - (-2)^{-2} \right) dx.$$

Using integration,

$$f(x) = \left[\left(\frac{1}{2}x - 3\right)^{-1} \cdot \frac{-1}{\frac{1}{2}} \right] - [(-2)^{-1} \cdot x] + c.$$

Substitute $x = 2$, $y = 3\frac{1}{2}$ to find c :

$$3\frac{1}{2} = 1 - \frac{1}{2} + c.$$

$$c = 3.$$

$$\text{Thus, } f(x) = \frac{-2}{\left(\frac{1}{2}x - 3\right)} \cdot \frac{x}{4} + 3.$$

(c) Set $f'(x) = 0$ to find the other stationary point:

$$\left(\frac{1}{2}x - 3\right)^{-2} - (-2)^{-2} = 0.$$

$$\left(\frac{1}{2}x - 3\right)^{-2} = 4.$$

$$\frac{1}{2}x - 3 = \pm 2.$$

Solving gives $x = 10$.

Substitute $x = 10$ into $f(x)$:

$$y = -1 - 2\frac{1}{2} + 3 = -\frac{1}{2}.$$

Check the nature using $f''(10)$:

$$f''(10) = -(5 - 3)^{-3} < 0, \text{ indicating a maximum.}$$

Solution to Question 53

June 2025 p12 q1

1. Apply a vertical stretch by a factor of 2. This transformation changes the graph of $y = f(x)$ to $y = 2f(x)$, effectively doubling the y-values of all points on the graph.

2. Translate the graph vertically by $\begin{pmatrix} 0 \\ -14 \end{pmatrix}$. This moves the graph down by 14 units, resulting in the graph of $y = g(x)$.

Solution to Question 54

June 2025 p12 q10

(a)(i) For an arithmetic progression, the difference between consecutive terms is constant. Thus, $k^2 - 4k = 8k - k^2$. Solving gives:

$$2k^2 = 12k$$

$$k(k - 6) = 0$$

Since k is non-zero, $k = 6$.

(a)(ii) The first term $a = 4k = 24$ and the common difference $d = k^2 - 4k = 12$. The sum of the first 20 terms is:

$$S_{20} = \frac{20}{2}(2 \times 24 + 19 \times 12) = 2760$$

(b) Let the first term be a and the common ratio be r . Then $ar^3 = 36$ and $ar^5 = 6$. Dividing gives:

$$r^2 = \frac{1}{6}$$

$$r = \frac{\sqrt{6}}{6}$$

$$a = \frac{36}{r^3} = \frac{1296}{\sqrt{6}}$$

The sum to infinity is:

$$S_{\infty} = \frac{a}{1-r} = \frac{1296}{\sqrt{6}-1}$$

Solution to Question 55

June 2025 p12 q11

(a) To express $x^2 + 4x + 2$ in the form $(x + a)^2 + b$, complete the square:

$$x^2 + 4x + 2 = (x + 2)^2 - 4 + 2 = (x + 2)^2 - 2.$$

Thus, $a = 2$ and $b = -2$.

(b)(i) To find $f^{-1}(x)$, start with $y = (x + 2)^2 - 2$.

Rearrange to solve for x :

$$y + 2 = (x + 2)^2$$

$$x = \pm\sqrt{y + 2} - 2$$

Since $x \leq -2$, choose the negative root:

$$f^{-1}(x) = -\sqrt{x + 2} - 2.$$

(b)(ii) To find $(gf)^{-1}(x)$, first find $gf(x)$:

$$gf(x) = g(f(x)) = g((x + 2)^2 - 2) = -(x + 2)^2 - 2 - 4 = -(x + 2)^2 - 6.$$

Now, solve $y = -(x + 2)^2 - 6$ for x :

$$y + 6 = -(x + 2)^2$$

$$-(x + 2)^2 = y + 6$$

$$x + 2 = \pm\sqrt{-y - 6}$$

Since $x \leq -2$, choose the negative root:

$$(gf)^{-1}(x) = -\sqrt{-x - 2} - 2.$$

Solution to Question 56

June 2025 p12 q2

First, solve the linear equation $2x + y + 4 = 0$ for y :

$$y = -2x - 4.$$

Substitute $y = -2x - 4$ into the curve equation $2xy + 5y^2 = 24$:

$$2x(-2x - 4) + 5(-2x - 4)^2 = 24.$$

Simplify and expand:

$$-4x^2 - 8x + 5(4x^2 + 16x + 16) = 24.$$

$$-4x^2 - 8x + 20x^2 + 80x + 80 = 24.$$

Combine like terms:

$$16x^2 + 72x + 56 = 0.$$

Divide the entire equation by 8:

$$2x^2 + 9x + 7 = 0.$$

Factor the quadratic equation:

$$(2x + 7)(x + 1) = 0.$$

Set each factor to zero and solve for x :

$$2x + 7 = 0 \Rightarrow x = -\frac{7}{2}.$$

$$x + 1 = 0 \Rightarrow x = -1.$$

Substitute $x = -\frac{7}{2}$ into $y = -2x - 4$:

$$y = -2\left(-\frac{7}{2}\right) - 4 = 3.$$

Substitute $x = -1$ into $y = -2x - 4$:

$$y = -2(-1) - 4 = -2.$$

Thus, the points of intersection are $(-1, -2)$ and $(-\frac{7}{2}, 3)$.

Solution to Question 57

June 2025 p12 q3

To find the coefficient of x^7 in the expansion of $\left(px^2 + \frac{4}{p}x\right)^5$, we use the binomial theorem. The general term in the expansion is given by:

$$\binom{5}{r}(px^2)^{5-r}\left(\frac{4}{p}x\right)^r$$

We need the power of x to be 7, so:

$$2(5-r) + r = 7$$

$$10 - 2r + r = 7$$

$$10 - r = 7$$

$$r = 3$$

Substitute $r = 3$ into the general term:

$$\binom{5}{3}(px^2)^{5-3}\left(\frac{4}{p}x\right)^3$$

$$= \binom{5}{3}(px^2)^2\left(\frac{4}{p}x\right)^3$$

$$= \binom{5}{3}p^2x^4 \cdot \frac{64}{p^3}x^3$$

$$= \binom{5}{3} \cdot \frac{64p^2}{p^3}x^7$$

$$= 10 \cdot \frac{64}{p}x^7$$

The coefficient of x^7 is $\frac{640}{p}$, and we know this equals 1280:

$$\frac{640}{p} = 1280$$

$$640 = 1280p$$

$$p = \frac{640}{1280} = \frac{1}{2}$$

Solution to Question 58

June 2025 p12 q4

(a) Differentiate $y = ax^{\frac{3}{2}} - 12x$ with respect to x :

$$\frac{dy}{dx} = \frac{3}{2}ax^{\frac{1}{2}} - 12.$$

Using the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Substitute $x = 9$ and $\frac{dx}{dt} = 5$:

$$\frac{dy}{dt} = \left(\frac{3}{2}a \times 9^{\frac{1}{2}} - 12\right) \times 5.$$

$$\frac{dy}{dt} = 5\left(\frac{9}{2}a - 12\right) = \frac{45}{2}a - 60 = \frac{15}{2}(3a - 8).$$

(b) For a minimum point at $x = \frac{1}{4}$, set $\frac{dy}{dx} = 0$:

$$\frac{3}{2}a\left(\frac{1}{4}\right)^{\frac{1}{2}} - 12 = 0.$$

$$\frac{3}{2}a \times \frac{1}{2} - 12 = 0.$$

$$\frac{3}{4}a = 12.$$

$$a = 16.$$

Solution to Question 59

P12 June q1

Integrate $\frac{dy}{dx}$:

$$y = \int (2x - 6x^{\frac{1}{2}}) dx.$$

$$y = x^2 - 4x^{\frac{3}{2}} + C.$$

Use the point (4, -9):

$$-9 = 4^2 - 4(4^{\frac{3}{2}}) + C.$$

$$-9 = 16 - 4(8) + C = -16 + C.$$

So $C = 7$.

Answer: $y = x^2 - 4x^{\frac{3}{2}} + 7$.

Solution to Question 60

P12 June q2

(a) Since $f(4 - x) = f(-(x - 4))$, the graph may be transformed by:

reflection in the y -axis, followed by a translation 4 units in the positive x -direction.

(b) A stretch with scale factor $\frac{1}{2}$ in the x -direction changes x to $2x$:

$$y = (2x)^3 - 3(2x) - 4 = 8x^3 - 6x - 4.$$

Then translate down by 3:

$$y = 8x^3 - 6x - 4 - 3.$$

Answer: $y = 8x^3 - 6x - 7$.

Solution to Question 61

P12 June q3

At points of intersection,

$$kx^2 - 5x - 6 = 3x - 7k.$$

So

$$kx^2 - 8x + 7k - 6 = 0.$$

For the line to intersect the curve, this equation must have real roots, so the discriminant is non-negative:

$$(-8)^2 - 4(k)(7k - 6) \geq 0.$$

$$64 - 28k^2 + 24k \geq 0.$$

$$7k^2 - 6k - 16 \leq 0.$$

Factorise:

$$(7k + 8)(k - 2) \leq 0.$$

Answer: $-\frac{8}{7} \leq k \leq 2$.

Solution to Question 62

P12 June q4

The coefficient of x^2 in $(2 - qx)^4$ is

$$\binom{4}{2}(2)^2(-q)^2 = 24q^2.$$

The coefficient of x^2 in $\left(1 + \frac{8}{q}x\right)^6$ is

$$\binom{6}{2}\left(\frac{8}{q}\right)^2 = \frac{960}{q^2}.$$

Since the second expansion is subtracted,

$$24q^2 - \frac{960}{q^2} = 324.$$

Multiply by q^2 :

$$24q^4 - 960 - 324q^2 = 0.$$

Divide by 12:

$$2q^4 - 27q^2 - 80 = 0.$$

Let $u = q^2$. Then

$$2u^2 - 27u - 80 = 0.$$

$$(2u + 5)(u - 16) = 0.$$

Since $u = q^2$, $u = 16$.

Answer: $q = 4$ or $q = -4$.