

Right triangles

Years 11–13 | Expanded international edition

Eighteen competition problems and three original bridge exercises. Give exact answers and justify your claims. Write your solutions on separate paper; hints and explained solutions are supplied separately.

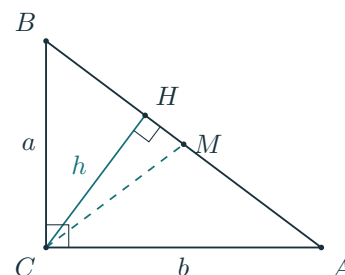
Toolkit The facts you will use

Here $\angle C = 90^\circ$, $CH \perp AB$, and M is the midpoint of AB . Set

$$a = BC, \quad b = CA, \quad c = AB,$$

$$h = CH, \quad p = AH, \quad q = HB.$$

The triangles ABC , ACH and CBH are similar. This explains the altitude and projection formulas.



Pythagoras	$a^2 + b^2 = c^2$
Area and altitude	$S = ab/2 = ch/2, \quad h = ab/c$
Projections	$p + q = c, \quad h^2 = pq, \quad b^2 = cp, \quad a^2 = cq$
Hypotenuse midpoint	$MA = MB = MC = c/2$
Inradius	$r = (a + b - c)/2 = ab/(a + b + c)$
Similarity	Lengths scale by k ; areas scale by k^2 .

Four circle facts

1. An angle subtended by a diameter is a right angle. The converse puts the right-angle vertex on the circle with the opposite side as diameter.
2. Two tangents from one exterior point have equal lengths. The radius at a tangency point is perpendicular to the tangent.
3. A tangent–chord angle equals the angle in the opposite segment.
4. Power of a point: $ZA \cdot ZB = ZC \cdot ZD$ for two secants; $ZT^2 = ZA \cdot ZB$ for a tangent. Equal secant products also give a converse for proving concyclicity. Intersecting chords have equal products too.

Three useful additions

Coordinates: place the right-angle vertex at $(0,0)$, with the legs on the axes. An incentre is equally distant from its three side lines.

Cosine rule: if sides u, v include angle γ and w is opposite it, then $w^2 = u^2 + v^2 - 2uv \cos \gamma$. In US2, use $\cos 135^\circ = -\sqrt{2}/2$.

Simple bounds: $xy \leq (x + y)^2/4$ for non-negative x, y , and $(s + t)^2 \leq 2(s^2 + t^2)$ for all real s, t . Both follow from a non-negative square.

Your route Five rungs, increasing in difficulty

The worksheet is a bank for several lessons. Complete a few problems carefully rather than trying to finish the whole bank in one sitting. UK, US, CN and KR identify the international additions; original 7.2.x numbers are retained.

1. **Build confidence** W1 → UK1 → 7.2.1 → UK2 → 7.2.9
2. **Connect ideas** W2 → UK3 → 7.2.2 → 7.2.3 → US1
3. **Find structure** 7.2.10 → W3 → 7.2.8 → CN1 → KR1
4. **Complete a proof** US2 → CN2 → 7.2.7 → 7.2.6 → 7.2.4
5. **IMO extension** KR2: a guided challenge after the earlier rungs

Rung 1 Build confidence

W1 A useful right triangle

Original bridge exercise; preparation for 7.2.1

A right triangle has legs of lengths 6 and 8. Find (a) its hypotenuse, (b) its area, (c) the altitude to the hypotenuse, and (d) the median to the hypotenuse. Explain which formula you use each time.

UK1 Area and perimeter reveal the sides

United Kingdom: UKMT Maclaurin Olympiad, 2019, Problem 2

Determine the three side lengths of a right triangle whose area is 150 cm^2 and whose boundary has total length 60 cm.

7.2.1 Three sticks

All-Russian Olympiad, 2021, school stage, Grade 11, Problem 4

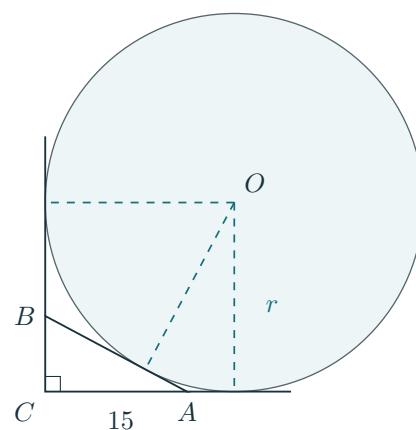
Senya has three straight sticks, each 24 cm long. He breaks one stick into two pieces. Using these two pieces and the two unbroken sticks, he forms the boundary of a right triangle. What is the area of the triangle, in square centimetres?

UK2 A circle outside the triangle

United Kingdom: UKMT Hamilton Olympiad, 2019, Problem 5

A right triangle has area 60 cm^2 ; one of its legs measures 15 cm. A circle touches the hypotenuse and the extensions of both legs, as in the figure. Its centre is on the opposite side of the hypotenuse from the right-angle vertex.

Calculate the circle's radius.



The circle touches the extended legs.

Rung 1, continued Turn a picture into a proof

7.2.9 The Pythagorean garden

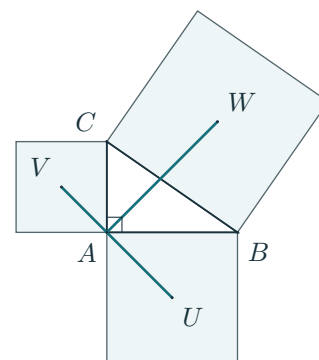
Nadezhda Energetiki (Energy Hope), 2016, Grade 11, Problem 5

Three paths form a right triangle with an acute angle α . Squares are then constructed externally on all three sides.

Two further straight paths are built: one joins the right-angle vertex to the centre of the largest square; the other joins the centres of the two smaller squares.

Which of these two new paths is longer? For which value of α is the difference between their lengths greatest?

In the diagram, the two paths are AW and UV .



All three squares lie outside the triangle.

Rung 2 Connect ideas

W2 A diameter and a perpendicular

Original bridge exercise; preparation for 7.2.2

A point P lies on the semicircle with diameter BC . The perpendicular from P meets BC at H , where $BH = 9$ and $HC = 16$. Find PH , PB and PC . Explain why a right-triangle altitude formula applies.

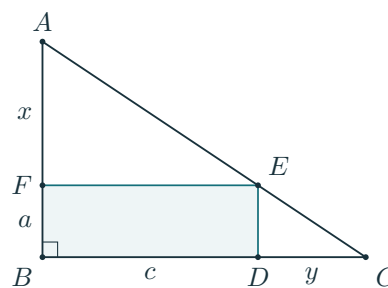
UK3 A rectangle with integer dimensions

United Kingdom: UKMT Mathematical Olympiad for Girls, 2024, Problem 1

Take a right triangle ABC , with $\angle B = 90^\circ$ and integer lengths AB and BC , in centimetres. A rectangle $BDEF$ is fitted inside it: D is on BC , F on BA , and E on AC . Set

$$BF = a, \quad AF = x, \quad BD = c, \quad DC = y.$$

- For $a = 2$, $c = 6$ and $y = 3$, determine x .
- Suppose the rectangle has area 9 cm^2 and integer side lengths. Determine every possible length of BC .



The two small triangles share the same slope.

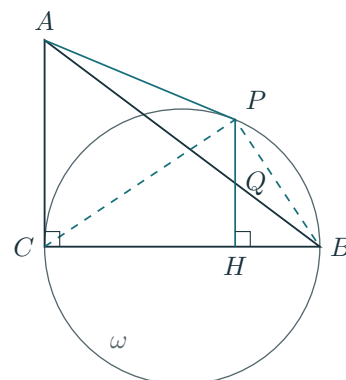
Rung 2, continued Combine tangents, similarity and area

7.2.2 A tangent creates a small triangle

Belchonok Olympiad, 2023, Grade 11, Problem 2

Triangle ABC is right-angled at C , with $BC = 26$. A circle ω is drawn with BC as diameter. From A , the tangent AP to ω is drawn, distinct from AC .

The perpendicular PH to BC meets AB at Q . Given that $BH : CH = 4 : 9$, find the area of triangle BPQ .



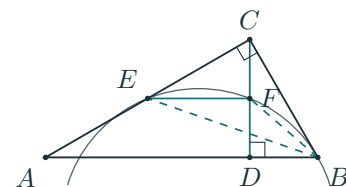
7.2.3 Parallel chords and a hidden midpoint

Phystech Olympiad, 2023, Grade 11, Problem 2

In triangle ABC , right-angled at C , the altitude to the hypotenuse is CD .

A circle tangent to the line BC at B meets CD at F and AC at E . It is known that $EF \parallel AB$ and $AD : DB = 3 : 1$.

Find $\frac{[ABC]}{[CEF]}$, where brackets denote area.



Only the relevant arc is shown.

US1 An unexpected area

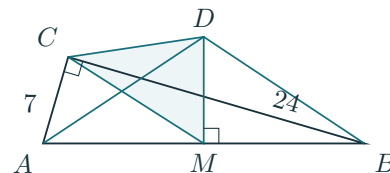
United States: AIME II, 2003, Problem 11; olympiad qualification competition

In a triangle with $\angle C = 90^\circ$, let $AC = 7$ and $BC = 24$. Mark the midpoint M of AB . Choose D in the same half-plane bounded by AB as C , with $AD = BD = 15$.

Write the area of CDM as

$$\frac{m\sqrt{n}}{p},$$

where m, n, p are positive integers, $\gcd(m, p) = 1$, and n has no prime-square factor. Find $m + n + p$.



Rung 3 Find structure

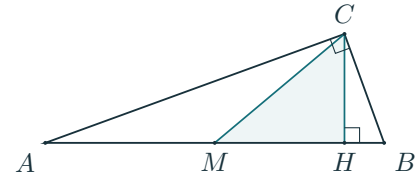
7.2.10 A repeated construction

Nadezhda Energetiki (Energy Hope), 2015, Grade 11, Problem 6

A park is a right triangle with hypotenuse 640 m and an acute angle $\alpha = \frac{11\pi}{24}$. From its right-angle vertex, draw the altitude and the median to the hypotenuse. These two segments and the part of the hypotenuse between their endpoints form a new right triangle. Repeat the construction in each new triangle.

Call the original triangle T_0 , and call the triangles formed after one, two, ... constructions $T_1, T_2, \dots$

- Find the hypotenuse of T_5 .
- Find the area of T_5 .



One step: the new triangle is CHM .
Schematic diagram.

W3 The converse of a circle theorem

Original bridge exercise; preparation for 7.2.8

Two non-collinear rays start at a point Z . Points U, V lie in that order on one ray, and W, X in that order on the other. Suppose that

$$ZU = 3, \quad ZV = 8, \quad ZW = 4, \quad ZX = 6.$$

Prove that U, V, W, X lie on one circle. State the converse theorem you use, or prove the claim by first drawing the circle through U, W, X .

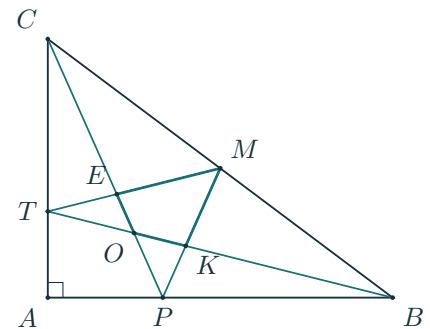
7.2.8 Four intersections, one circle

All-Siberian Olympiad, 2019, Grade 11, Problem 4

In triangle ABC , right-angled at A , let M be the midpoint of the hypotenuse BC . Points P and T lie on AB and AC , respectively, with

$$AP : PB = AT : TC = 1 : 2.$$

Let $K = BT \cap PM$, $E = CP \cap MT$, and $O = CP \cap BT$.
Prove that quadrilateral $OKME$ is cyclic.



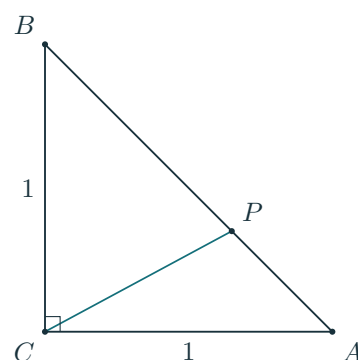
Rung 3, continued Optimise and construct

CN1 Maximise a product of three distances

China: Western Mathematical Olympiad, 2005, Problem 6

A point P moves along the boundary of triangle ABC , where $CA = CB = 1$ and $\angle ACB = 90^\circ$.

Find the greatest possible value of $PA \cdot PB \cdot PC$, and identify every position of P at which it is attained.



P may lie on any of the three sides.

KR1 Pythagorean areas without large gaps

Republic of Korea: final selection examination, 1993, Problem 4

Call a positive integer a *Pythagorean area* if it is the area of a right triangle whose three side lengths are positive integers.

For each integer $n > 12$, prove that a Pythagorean area N exists with

$$n < N < 2n.$$

Rung 4 Complete a proof

US2 Can every marked length be an integer?

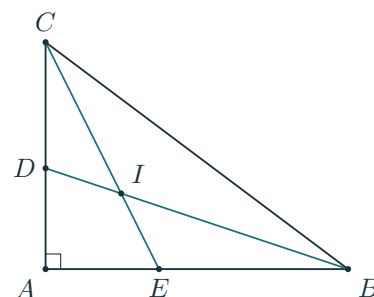
United States: USAMO, 2010, Problem 4

In a right triangle ABC , the right angle is at A . The internal bisectors from B and C meet AC and AB at D and E , respectively. Let $I = BD \cap CE$.

Is there a triangle for which all six lengths

$$AB, AC, BI, ID, CI, IE$$

are integers? Give a proof of your conclusion.



CN2 Cover an obtuse triangle

China: Girls' Mathematical Olympiad, 2004, Problem 3

An obtuse triangle has circumradius 1. Show that it can be placed entirely inside an isosceles right triangle with hypotenuse $1 + \sqrt{2}$. Points on the boundary count as covered.

Rung 4, continued Longer geometric arguments

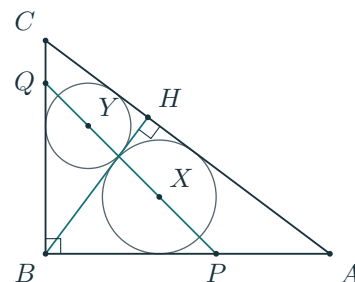
7.2.7 Two incentres and a fixed area

Belchonok Olympiad, 2021, Grade 11, Problem 4

In triangle ABC , right-angled at B , the altitude to the hypotenuse AC is BH .

Let X and Y be the incentres of triangles ABH and CBH , respectively. The line XY meets AB at P and BC at Q .

Find the area of triangle BPQ , given that $BH = a$.



7.2.6 A tangent angle bisector

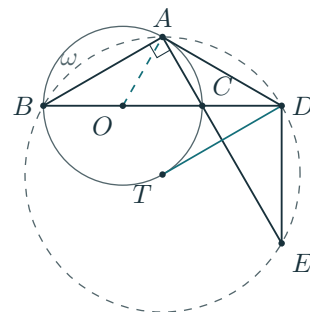
Belchonok Olympiad, 2018, Grade 11, Problem 4

Triangle ABC is right-angled at A , with hypotenuse BC and circumcircle ω .

A point D is chosen on the extension of BC so that AD is tangent to ω . The line AC meets the circumcircle of triangle ABD again at $E \neq A$.

The internal bisector of $\angle ADE$ is also tangent to ω . In what ratio does C divide the segment AE ? Give $AC : CE$.

The diagram also labels the centre O of ω and the second point of tangency T .



The dashed circle passes through A, B, D, E .

Rung 4, continued Check every configuration

7.2.4 One condition, more than one configuration

Step into the Future Olympiad, 2022; listed as Grade 11, Problem 3 in the supplied collection

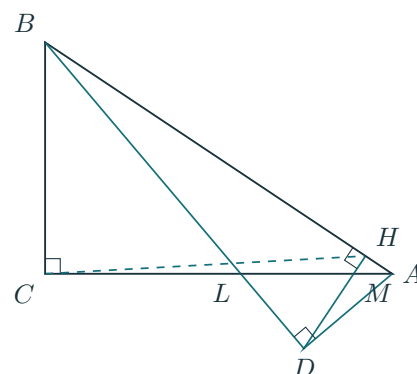
Triangle ABC is right-angled at C . Point M lies on AC , with $AM = 2$ and $MC = 16$. The altitude of triangle AMB from M meets AB at H .

Point D lies on the line MH , with $\angle ADB = 90^\circ$, and C, D lie on the same side of the line AB .

Let $L = BD \cap AC$. If

$$\tan \angle ACH = \frac{1}{18},$$

find all possible values of BL .



One possible configuration.

Rung 5 IMO extension

KR2 A diameter, an arc midpoint and an incentre

IMO 2002, Problem 2; proposed by Hojoo Lee, Republic of Korea

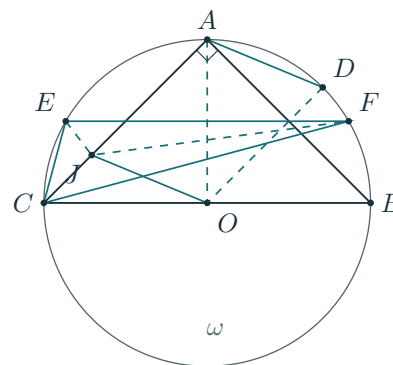
Let a circle ω have centre O and diameter BC . Select a point A on it with

$$0^\circ < \angle AOB < 120^\circ.$$

Bisect the arc from A to B that avoids C , marking its midpoint D . The parallel to AD through O crosses the line AC at J .

Let E, F be the two intersections of ω with the perpendicular bisector of OA .

Prove that J is the incentre of CEF .



Triangle ABC is right-angled at A .

For the highest rung. Try a construction or a useful equal-length statement first. A complete proof must also establish where J lies; a diagram alone does not do this.

Sources Contest, year and problem number

The international questions are classroom restatements in English. Their conditions and numerical data match the cited competition problems; diagrams and expanded explanations were prepared for this pack. The extra request to identify equality cases in CN1 makes its maximum-value answer precise.

- **UK1, UK2.** UKMT, *Intermediate Mathematical Olympiad solutions*, 2019: Maclaurin Problem 2 (PDF p. 15); Hamilton Problem 5 (PDF p. 12). [Official UKMT paper and solutions](#).
- **UK3.** UKMT, *Mathematical Olympiad for Girls*, 2024, Problem 1 (PDF p. 5). [Official report and problem paper](#).
- **US1.** AIME II, 2003, Problem 11. [MAA-licensed problem reprint](#), LIVE by Po-Shen Loh. AIME is a qualification competition, not the USAMO proof examination.
- **US2.** USAMO, 2010, Problem 4. [Original MAA Day II paper](#), [archived copy](#).
- **CN1, CN2.** Western Mathematical Olympiad 2005, Problem 6; Girls' Mathematical Olympiad 2004, Problem 3. In *Mathematical Olympiad in China: Problems and Solutions*, by the Chinese IMO National Coaches Team, edited by Xiong Bin and Lee Peng Yee (2007), printed pp. 200–201 and 144–145 respectively. [Organizer-authored collection](#), [archived PDF](#).
- **KR1.** Republic of Korea, final selection examination, 18 April 1993, Problem 4. [Archived Korean Mathematical Society source](#); [English problem-paper transcription](#).
- **KR2.** IMO, 2002, Problem 2. [Official problem paper](#). The equivalent angle condition $0^\circ < \angle AOB < 120^\circ$ is explicit in [Hojoo Lee's problem collection](#); the attribution to Hojoo Lee (KOR) is recorded in [Evan Chen's IMO notes](#). KR2 is a Korean-proposed international problem, not a KMO paper problem.

Original selection. The nine 7.2.x problems are translations of the selection supplied for this lesson: I. V. Yakovlev, *Olympiad Mathematics, Grade 11* (2023), [Section 7.2](#), pp. 70–71. The supplied selection has no 7.2.5. W1–W3 are original bridge exercises. In 7.2.10, T_0 denotes the original triangle.