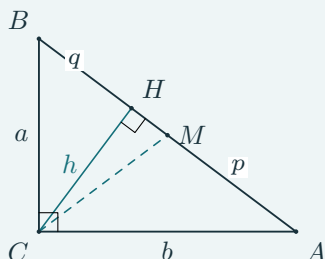


Right triangles

Year 10–11 · Four lessons and six further challenges

How to work. Use separate paper. Give exact values and explain the geometric facts you use. The numbered tasks are genuine contest problems, with their original sources identified.

1. Pythagoras and area



$\angle C = 90^\circ$, $BC = a$, $CA = b$, $AB = c$.

H is the foot of the altitude; M is the hypotenuse midpoint.

Set $AH = p$, $HB = q$, $CH = h$.

$$c^2 = a^2 + b^2, \quad S = \frac{ab}{2} = \frac{ch}{2}.$$

$$h = \frac{ab}{c}, \quad CM = \frac{c}{2}.$$

The converse is useful: if the largest side c satisfies $c^2 = a^2 + b^2$, the triangle is right-angled.

2. Similarity and projection

Two equal angles are enough to prove triangles similar. Match corresponding sides before taking ratios. Lengths scale by k ; areas scale by k^2 .

$$p + q = c, \quad a^2 = cq, \quad b^2 = cp, \quad h^2 = pq.$$

These formulas follow from the three similar right triangles made by the altitude. Triangles sharing a perpendicular height have areas proportional to their bases.

3. Circles, tangents and angles

An angle subtended by a diameter is 90° . A radius to a tangent is perpendicular to it. Tangent segments from the same external point are equal. At a point where two circles touch, both centres and the contact point lie on one line.

External tangency: $OO' = R + r$; internal tangency: $OO' = R - r$ ($R > r$).

Circle area is πr^2 ; a sector of angle θ° has area $\frac{\theta}{360} \pi r^2$.

Useful radius formulas. For a right triangle with legs a, b and hypotenuse c ,

$$r_{\text{in}} = \frac{a + b - c}{2}, \quad r_{\text{ex}} = \frac{a + b + c}{2} \quad (\text{excircle opposite the right angle}).$$

For any triangle with semiperimeter s , its area is $S = r_{\text{in}} s$.

The ladder. Lesson 1: recognise $\rightarrow$ Lesson 2: connect $\rightarrow$ Lesson 3: construct $\rightarrow$ Lesson 4: justify. In each lesson, work in order. The final tasks are stretch work; the further challenges are optional.

Diagrams are precise illustrations; use the stated conditions, not measurements. Multiple-choice options have been omitted. Short English classroom restatements retain the original mathematical data.

Lesson 1 Find the right triangle

Problems 1–6 · Pythagoras, displacement and hidden lengths

1 Radical legs

UK · UKMT Intermediate Mathematical Challenge 2024, Q4

Calculate the hypotenuse of a right triangle whose legs measure $\sqrt{5}$ and $\sqrt{12}$.

2 A runner's return

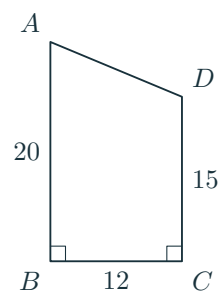
USA · AMC 10A 2010, Q7

A runner covers one mile north, then one mile north-east, then one mile south-east. Find the length of the straight journey back to the starting point.

3 A trapezium with a hidden triangle

Canada · CEMC Cayley Contest 2024, Q13

In the pictured quadrilateral, $AB = 20$, $BC = 12$, $CD = 15$ and both AB and CD are perpendicular to BC . Determine its perimeter.



4 Four points on one circle

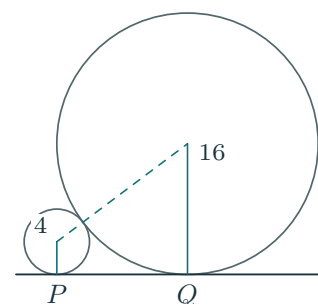
UK · UKMT Intermediate Mathematical Challenge 2024, Q6

Four of $(5, 0)$, $(4, 3)$, $(2, 2)$, $(3, 4)$ and $(0, 5)$ belong to the same circle. Identify the remaining point.

5 Touching spheres

UK · UKMT Intermediate Mathematical Challenge 2025, Q22

Two spheres, with radii 4 cm and 16 cm, rest on a horizontal surface and touch each other. How far apart are their points of contact with the surface?



Cross-section through both centres.

6 Three sticks

Russia · All-Russian Olympiad 2021, school stage, Grade 11, Q4; supplied selection 7.2.1

Three straight sticks each have length 24 cm. One is broken into two pieces. Using both pieces and both unbroken sticks, with no overlap, a right triangle's boundary is formed. Find its area.

Lesson 2 Use similarity and area

Problems 7–12 · Two area formulas, projection and folds

7 Area from two totals

UK · UKMT Intermediate Mathematical Challenge 2023, Q16

A right triangle's side lengths total 16 cm, and their squares total 98 cm^2 . Find the triangle's area.

8 Area and perimeter reveal the sides

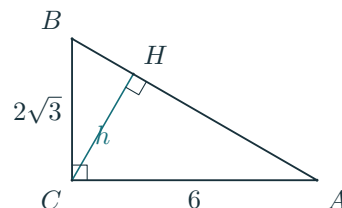
UK · UKMT Maclaurin Olympiad 2019, Q2

Determine all three sides of a right triangle with area 150 cm^2 and perimeter 60 cm.

9 The third altitude

USA · AMC 10A 2014, Q9

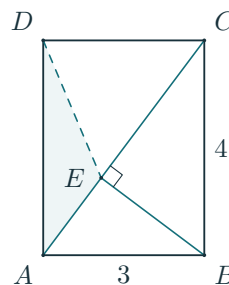
Two altitudes of a right triangle are its legs, of lengths $2\sqrt{3}$ and 6. Calculate the remaining altitude.



10 An altitude inside a rectangle

USA · AMC 10B 2017, Q15

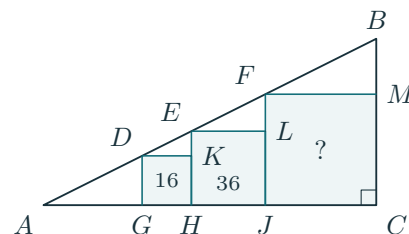
Rectangle $ABCD$ has $AB = 3$ and $BC = 4$. Point E is the perpendicular projection of B onto AC . Find the area of triangle AED .



11 Three growing squares

Canada · CEMC Cayley Contest 2024, Q19

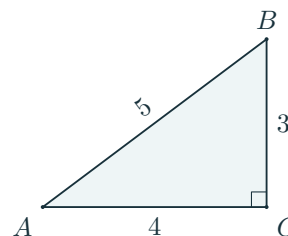
Triangle ABC is right-angled at C . The adjacent squares $DKHG$, $ELJH$ and $FMCJ$ sit on AC , as shown, with D, E, F on AB and M on BC . The first two square areas are 16 and 36. Find the third square's area.



12 A fold in a 3–4–5 triangle

USA · AMC 10A 2018, Q13

A triangular sheet has $AC = 4$, $BC = 3$, $AB = 5$ inches. It is folded so that A lands exactly on B . Determine the length of the crease.



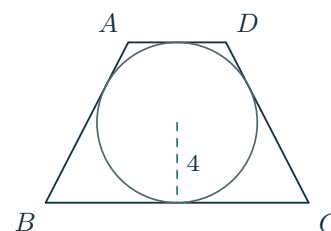
Lesson 3 Circles create right triangles

Problems 13–18 · Radii, common chords and tangency

13 An inscribed circle in a trapezium

UK · UKMT Hamilton Olympiad 2018, Q4

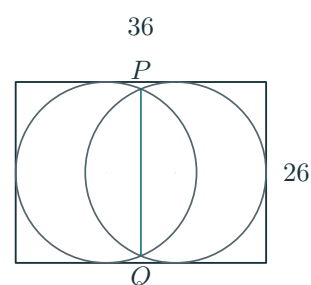
An isosceles trapezium $ABCD$, with $AD \parallel BC$ and $AB = CD$, is tangent to a circle along all four sides. The radius is 4 and the trapezium's area is 72. Find AB .



14 A common chord

UK · UKMT Intermediate Mathematical Challenge 2025, Q18

A 36 cm by 26 cm rectangle contains two circles, each tangent to the top and bottom sides and to a different vertical side. Their intersection points are P, Q . Find PQ .



15 Distances to a rectangle's corners

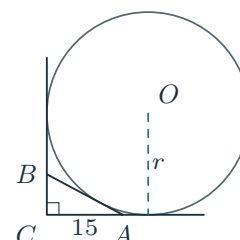
UK · UKMT Intermediate Mathematical Challenge 2025, Q23

A point P inside a rectangle is 5 cm from one corner and 14 cm from the opposite corner. A third corner is 10 cm from P . Find the distance to the fourth corner.

16 A circle outside a right triangle

UK · UKMT Hamilton Olympiad 2019, Q5

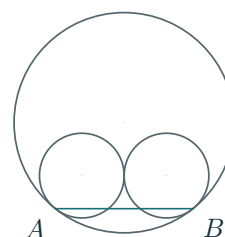
A right triangle has area 60 cm^2 and one leg of length 15 cm. A circle touches its hypotenuse and the extensions of both legs beyond their other endpoints, as pictured. Find the circle's radius.



17 Two circles inside a larger circle

USA · AMC 10A 2018, Q15

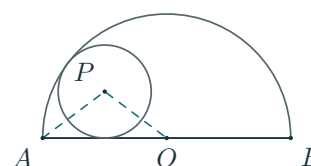
Two circles of radius 5 touch externally and each touches a circle of radius 13 internally, at A and B . Write $AB = m/n$ in lowest terms, with positive integers m, n , and determine $m + n$.



18 A circle inside a semicircle

UK · UKMT Intermediate Mathematical Challenge 2024, Q23

A circle with centre P is tangent to a semicircle of centre O and diameter AB , and to AB . Given $PA = PO$, find the ratio of its radius to the semicircle's radius.



Lesson 4 Write an olympiad argument

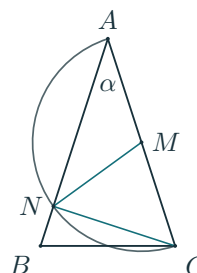
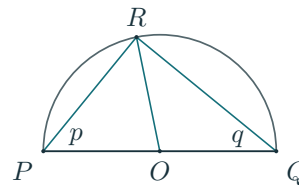
Problems 19–24 · Circle angles, area identities and complete cases

19 Prove a right angle; then chase angles

UK · UKMT Mathematical Olympiad for Girls 2024, Q3

(a) O is the centre of a semicircle of diameter PQ , and R lies on its arc. Set $p = \angle RPQ$, $q = \angle PQR$. Express angles using p, q to prove $\angle PRQ = 90^\circ$.

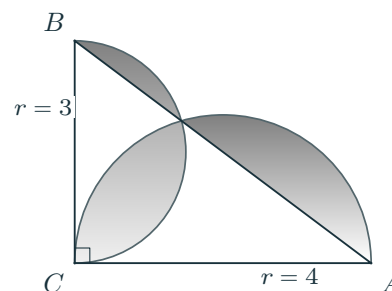
(b) $AB = AC$. The semicircle with diameter AC and centre M meets AB again at N . If $\angle AMN = 6\angle BCN$, find $\alpha = \angle BAC$. Give geometric reasons.



20 Two semicircles and a right triangle

UK · UKMT Intermediate Mathematical Challenge 2023, Q21

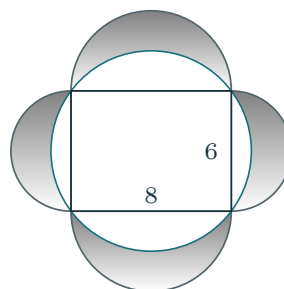
Semicircles of radii 3 and 4 are drawn inward on the two legs of a right triangle, as pictured. Find the shaded area: the parts outside the triangle together with the overlap inside it.



21 Four lunes around a rectangle

Canada · CEMC Cayley Contest 2024, Q23

Four outward semicircles have the sides of a 6 by 8 rectangle as diameters. A circle passes through its four vertices. Let A be the area inside the semicircles but outside this circle. Find the integer closest to A .



Lesson 4, continued Complete the argument

Problems 22–24 · Integer cases, a hidden angle and six squares

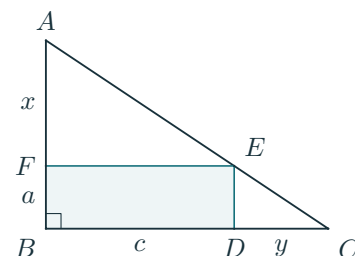
22 An integer rectangle in a right triangle

UK · UKMT Mathematical Olympiad for Girls 2024, Q1

Right triangle ABC has integer leg lengths AB, BC in centimetres. Rectangle $BDEF$ is inscribed as shown, with $BF = a$, $AF = x$, $BD = c$ and $DC = y$.

(a) If $a = 2$, $c = 6$, $y = 3$, determine x .

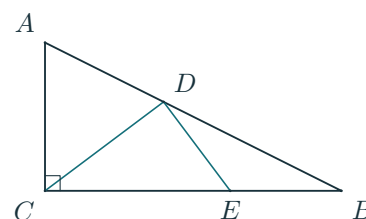
(b) The rectangle has area 9 cm^2 and integer dimensions. Find every possible value of BC .



23 A hidden right angle

USA · AMC 10B 2019, Q16

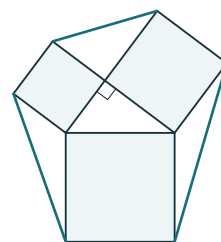
Triangle ABC is right-angled at C . Points $D \in AB$ and $E \in BC$ are interior to those sides. Given $AC = CD$, $DE = EB$ and $AC : DE = 4 : 3$, find $AD : DB$.



24 Six squares: a Pythagorean identity

UK · UKMT Maclaurin Olympiad 2018, Q4

Build outward squares on a scalene right triangle's sides. Join their neighbouring outer corners to form the pictured hexagon, then build outward squares on its three new connecting segments. Two of the six squares have equal areas totalling 2018 cm^2 . Find the sum of all six areas.



Build the extra squares on the three bold segments.

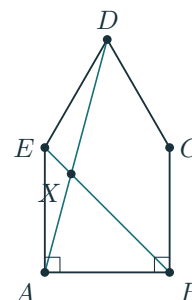
Further challenges Choose another route

Problems 25–30 · Additional proofs, counting and configurations

25 A pentagon made from a square and a roof

UK · UKMT Hamilton Olympiad 2019, Q2

Convex pentagon $ABCDE$ has five equal sides and right angles at A and B , as pictured. Lines AD and BE meet at X . Prove $DX = BX$.



26 Two right triangles share two lengths

USA · AMC 10B 2019, Q15

Two right triangles have areas 1 and 2. Two sides of the first match two sides of the second in length. Find the square of the product of their remaining side lengths.

27 How many integer-length segments?

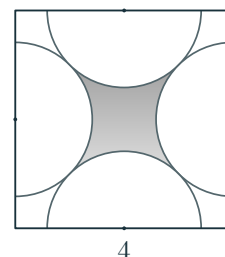
USA · AMC 10A 2018, Q16

Right triangle ABC has legs $AB = 20$ and $BC = 21$. Including the two legs, how many different segments of integer length join B to a point of the hypotenuse AC ?

28 Four tangent semicircles

UK · UKMT Intermediate Mathematical Challenge 2024, Q21

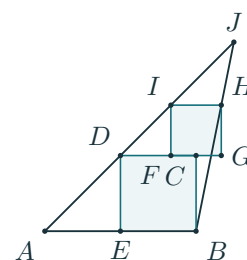
Four identical inward semicircles have centres at the side midpoints of a square of side 4 cm. Each touches its two neighbours. Calculate the shaded central area.



29 Two squares inside a triangle

UK · UKMT Maclaurin Olympiad 2019, Q4

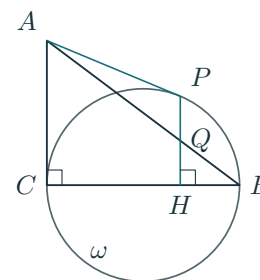
The pictured squares $BCDE$ and $FGHI$ fit in triangle ABJ . Point E is the midpoint of AB , and C is the midpoint of FG . Find the ratio $[BCDE] : [ABJ]$.



30 A tangent creates a small triangle

Russia · Belchonok Olympiad 2023, Grade 11, Q2; supplied selection 7.2.2

Triangle ABC is right-angled at C and $BC = 26$. Circle ω has diameter BC . The tangent from A other than AC touches it at P . The perpendicular PH to BC meets AB at Q . If $BH : CH = 4 : 9$, find $[BPQ]$.



Sources Contest provenance

Original question numbers are shown under every task.

Selection. 17 UK tasks, 8 USA tasks, 3 Canadian tasks and 2 Russian tasks. UKMT Hamilton, Maclaurin and MOG are olympiad papers. IMC, AMC 10 and Cayley are contest papers; their questions are used here as open-response preparation for proof olympiads. All explanations and redrawn figures in this pack were prepared for the club.

UKMT IMC 2023 · our 7, 20 (original Q16, Q21)

https://ukmt.org.uk/wp-content/uploads/2023/08/IMC_2023-Paper-1.pdf

UKMT IMC 2024 · our 1, 4, 18, 28 (original Q4, Q6, Q23, Q21)

https://ukmt.org.uk/wp-content/uploads/2024/02/IMC_2024-Paper.pdf

UKMT IMC 2025 · our 5, 14, 15 (original Q22, Q18, Q23)

<https://ukmt.org.uk/wp-content/uploads/2025/01/IMC-2025-Paper.pdf>

UKMT IMOK 2018 · our 13 (Hamilton Q4), 24 (Maclaurin Q4)

<https://archive.ukmt.org.uk/pdfs/SolnsAllIMOKOlympiads18.pdf>

UKMT IMOK 2019 · our 8 (M2), 16 (H5), 25 (H2), 29 (M4)

<https://archive.ukmt.org.uk/pdfs/SolnsAllIMOKOlympiads19.pdf>

UKMT MOG 2024 · our 19 (Q3), 22 (Q1)

<https://bmos.ukmt.org.uk/home/ukmog-2024.pdf>

CEMC Cayley 2024 · our 3, 11, 21 (original Q13, Q19, Q23)

<https://cemc.uwaterloo.ca/sites/default/files/documents/2024/2024CayleyContest.html>

USA · Mathematical Association of America

The original AMC questions are checked against the MAA-permitted reprints at LIVE by Po-Shen Loh. These reprints explicitly identify the MAA copyright and permission.

AMC 10A 2010, Q7 (our 2); AMC 10A 2014, Q9 (our 9); AMC 10B 2017, Q15 (our 10).

AMC 10A 2018, Q13, Q15, Q16 (our 12, 17, 27); AMC 10B 2019, Q16, Q15 (our 23, 26).

Russian selection supplied by the teacher · our 6 and 30

<https://mathus.ru/math/11math2023.pdf>

I. V. Yakovlev, *Olympiad Mathematics, Grade 11* (2023), Section 7.2, printed pp. 70–71. Our 6 is 7.2.1 (All-Russian Olympiad 2021, school stage, Grade 11, Q4); our 30 is 7.2.2 (Belchonok 2023, Grade 11, Q2). These are English translations of the supplied selection. Problem 30 is optional stretch work for this younger group.