

Quadratic equations: olympiad ladders

Student questions | 13 problems | Write your solutions on separate paper

Give full reasoning. A *monic* quadratic has leading coefficient 1. “Exactly one real root” means a repeated root. Problems 9-13 are optional extensions.

Ladder A Read the roots through the coefficients

1. A quick start

Preparatory exercise for this lesson

Let r, s be the roots of $x^2 - 7x + 10 = 0$. Without finding the roots individually, calculate

$$r + s, \quad rs, \quad r^2 + s^2, \quad (r - s)^2.$$

2. Keep both possibilities

SAMMAT 2023, Grade 8, Problem 10 | Your source: 4.1.2

The equation $x^2 + ax + 6 = 0$ has roots x_1, x_2 satisfying $x_1^2 + x_2^2 = 13$. Find every possible value of $x_1 + x_2$.

3. Two inputs, one output

Step into the Future 2020, Grade 8, Problem 2 | Your source: 4.1.6

For $f(x) = x^2 - 5x + 2020$, solve

$$f(3 - x) = f(3x - 1).$$

4. Doubling the roots

USA: AMC 10B 2005, Problem 16; also AMC 12B, Problem 12

The roots of $x^2 + mx + n = 0$ are obtained by doubling the roots of $x^2 + px + m = 0$. Assume m, n, p are all nonzero. Determine n/p .

Ladder B Reconstruct the missing information

5. Find a quadratic inside a triangle

UKMT Intermediate Mathematical Olympiad, Maclaurin 2019, Problem 2

Find the three side lengths of a right-angled triangle whose perimeter is 60 cm and whose area is 150 cm^2 .

6. Three numbers, four roles

Open Olympiad 2021, Grade 8, Problem 1 | Your source: 4.1.4

A monic equation $x^2 + px + q = 0$ has roots x_1, x_2 . Three of the four numbers p, q, x_1, x_2 are 1, 2, -6, but their roles are unknown. Find the fourth number.

7. One root in each equation

Open Olympiad 2022, Grade 8, Problem 1 | Your source: 4.1.3

Each of $x^2 + px + q = 0$ and $x^2 + ax + b = 0$ has exactly one real root. Three of p, q, a, b are 16, 64, 1024. Find all possibilities for the fourth number.

Quadratic equations: olympiad ladders

Student questions | continued

Ladder B Use the distance between the roots

8. A discriminant without an equation

Open Olympiad 2023, Grade 8, Problem 2 | Your source: 4.1.5

A monic quadratic with roots x_1, x_2 has discriminant 18; a monic quadratic with roots x_3, x_4 has discriminant 50. Find all possible discriminants of the monic quadratic whose roots are $x_1 + x_3$ and $x_2 + x_4$.

Ladder C Prove and extend [Optional problems 9-13]

9. Cube the coefficients

SAMMAT 2022, Grade 8, Problem 10 | Your source: 4.1.1

Let a, b, c be real, with $a \neq 0$. The polynomial $ax^2 + bx + c$ has real roots. Must $a^3x^2 + b^3x + c^3$ also have real roots? Justify your answer.

10. Compare two discriminants

Belchonok 2022, Grade 8, Problem 2 | Your source: 4.1.7

Suppose $p \neq 0$ and $px^2 + qx + r$ has two real roots, possibly equal. Prove that

$$3px^2 + 2(p+q)x + (q+r)$$

also has real roots. Show also that they are distinct if the original roots are distinct.

11. Integer roots give integer squares

UKMT Intermediate Mathematical Olympiad, Maclaurin 2022, Problem 5

Both equations $x^2 + bx + c = 0$ and $x^2 + bx - c = 0$ have two distinct integer roots, and b, c are nonzero.

(a) Prove that some distinct positive integers p, q satisfy $2b^2 = p^2 + q^2$.

(b) Prove that some distinct positive integers r, s satisfy $b^2 = r^2 + s^2$.

12. Count the possible quadratics

USA: AIME 1991, Problem 8

How many real parameter values a make both roots of

$$x^2 + ax + 6a = 0$$

integers? Repeated integer roots are allowed.

13. A quartic built from two quadratics

USA Mathematical Olympiad, 1984, Problem 1 | Final stretch

Two of the four roots of

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

have product -32 . Determine k .