

AM-GM: a ladder to olympiad proofs

Student questions | 12 problems | Work on separate paper

Give a proof for each claim. For inequalities, identify all equality cases. Problems marked *Extension* are optional; you are not expected to finish every problem in one lesson.

Ladder A Choose the terms, then multiply

1. A fixed product

MathUs handout, Problem 12; the case $n = 3$

Suppose $a, b, c > 0$ and $abc = 1$. Prove that

$$(1 + a)(1 + b)(1 + c) \geq 8.$$

2. A coefficient hides an extra term

MathUs handout, Problem 24

Find the minimum of $2x + \frac{1}{x^2}$ over all $x > 0$, and determine where it is attained.

3. Four terms, with a sign check [Extension]

MathUs handout, Problem 4

For arbitrary real numbers x and y , prove

$$x^4 + y^4 + 8 \geq 8xy.$$

4. Split, apply, multiply

Moscow Mathematical Olympiad, 2008 district round, Grade 10, Problem 5; MathUs Problem 11

For $x, y, z > 0$ with $xyz = 1$, establish

$$(2 + x)(2 + y)(2 + z) \geq 27.$$

Ladder B Turn bounds into exact answers

5. Six products and four variables

Bridge exercise prepared for this lesson

For $a, b, c, d > 0$, write $P = abcd$ and $T = ab + ac + ad + bc + bd + cd$.

(a) Prove that $T \geq 6\sqrt{P}$.

(b) If $a + b + c + d = 12$, prove that $\sqrt{P} \leq 9$.

6. Two bounds meet

British Mathematical Olympiad, Round 2, 1996, Problem 4

Determine every positive quadruple (a, b, c, d) satisfying

$$a + b + c + d = 12, \quad abcd = 27 + ab + ac + ad + bc + bd + cd.$$

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Student questions | continued

Ladder B Transfer the idea [Extensions]

7. A coefficient controlled by its roots

Moscow Mathematical Olympiad, 2016, Grade 10, Problem 3; MathUs Problem 30

The polynomial $x^4 + ax^3 + bx^2 + cx + d$ has integer coefficients and four positive real roots, counting multiplicity. What is the least possible value of b ? Prove that your value can occur.

8. Combine two upper bounds

British Mathematical Olympiad, Round 1, 2002/03, Problem 3; held 11 December 2002

Assume $x, y, z > 0$ and $x^2 + y^2 + z^2 = 1$. Establish

$$x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}.$$

Ladder C Reveal a hidden AM-GM inequality

9. A product hidden in three fractions

Bridge exercise prepared for this lesson

For $u, v, w > 0$, prove

$$\frac{vw}{u^2} + \frac{wu}{v^2} + \frac{uv}{w^2} \geq 3.$$

10. Build the algebraic bridge

Bridge exercise prepared for this lesson

For $a, b, c > 0$, let $S = a^2 + b^2 + c^2 + ab + bc + ca$.

(a) Verify the identity

$$\frac{2ab + S}{(a + b)^2} = 1 + \frac{(a + c)(b + c)}{(a + b)^2}.$$

(b) Assuming $S \leq 2$, deduce a lower bound of the same form for $\frac{2ab + 2}{(a + b)^2}$.

11. The USAMO target

USA Mathematical Olympiad, 2011, Problem 1

Suppose $a, b, c > 0$ satisfy $a^2 + b^2 + c^2 + (a + b + c)^2 \leq 4$. Show that

$$\frac{ab + 1}{(a + b)^2} + \frac{bc + 1}{(b + c)^2} + \frac{ca + 1}{(c + a)^2} \geq 3.$$

12. Choose larger expressions as the terms [Extension]

USA Junior Mathematical Olympiad, 2018, Problem 2

Assume $a, b, c > 0$ with $a + b + c = 4\sqrt[3]{abc}$. Establish

$$2(ab + bc + ca) + 4\min(a^2, b^2, c^2) \geq a^2 + b^2 + c^2.$$

Here $\min(a^2, b^2, c^2)$ denotes the smallest of the three squares.